Laws of material sciences, particularly those of physics, are quantitative. They are expressed as MATHEMATICAL relationships between physical quantities.

MATHEMATICS

In great scientists, a highly creative intellect often combines with outstanding MATHEMATICAL talent.

A TEXTBOOK FOR CLASS XI
PART II

MATHEMATICS

A Textbook for Class XI
Part II

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CHAPTER 8

CARTESIAN SYSTEM OF RECTANGULAR COORDINATES

8.1 Introduction

The geometry you have been studying thus far in earlier classes is called *Euclidean Geometry*. Its approach, as you would recall, was to start with certain *concepts*, like the concepts of points, lines and planes; attribute certain properties to them which we called *axioms* or *postulates* (suggested by physical experience); and then using the methods of deductive logic to derive a number of *theorems* which formed the main fruit of our mathematical activity and revealed to us the interesting and useful properties of the geometric figures under consideration. This approach was first presented by the Greek mathematician *Euclid*, around 300 BC, in his famous treatise "Elements" comprising thirteen books, and is being followed since then. As you would also recall, this made essentially no use of the processes of algebra, and is called the *synthetic* approach to geometry.

This was the only approach to geometry for some two thousand years till the French philosopher and mathematician Rene Descartes (1596-1665) published "La Geometrie" in 1637 wherein he introduced the analytic approach (as against synthetic) by systematically using algebra in his study of geometry. This was achieved by representing points in the plane by ordered pairs of real numbers (called *Cartesian Coordinates* named after Rene Descartes), and representing lines and curves by algebraic equations. This wedding of algebra and geometry is known as *analytic* or *coordinate geometry*, and this is what we propose to study here.

8.2 Cartesian Coordinate System—The Number Plane

In this section we shall establish a 1-1 correspondence between points on a straight line and the real numbers, and subsequently a 1-1 correspondence between points in the Euclidean plane and ordered pairs of real numbers. This would make it possible to apply the methods of algebra to study problems in geometry.

We are familiar with the representation of real numbers on a line, which we call the *real line*, or the number line, and denote by R′ (or R). This was achieved through directed line-segments and fixing a unit for length measurement. Fix a point O on the line, which we shall call the *origin* from where all distances should be measured. This divides the line into two parts, the points on the left and right of the origin O. The distances measured (in terms of the fixed units) in the two parts are taken to be of opposite signs. This gives us the idea of directed line segments where not only length, but directions are also taken into account. If A is any other point on the line, then the line segment OA will be called directed line segment, directed from O to A. Obviously then, as directed line segments, $\overrightarrow{OA} = -\overrightarrow{AO}$. Distances measured to the right are conventionally taken as positive, and those measured to the left as negative. Thus every point P on this line corresponds to a real number x whose magnitude is the length OP measured in the prescribed units, and whose signature (sign) is +ve or −ve according as P is to the right or left of the origin O. Conversely, given a real number x we can always find a point P on the line on the right or left of O depending on the sign of x, such that the length OP equals $|x|$ units. This establishes a 1-1 correspondence between the points on the line and real numbers.

We now proceed to define a 1-1 correspondence between the points in the Euclidean plane and the set of all *ordered pairs of real numbers* (a, b). This can be done by defining what is called a *Cartesian Coordinate System* on the Euclidean plane, which we do as under:

In the Euclidean plane draw a horizontal line $X' OX$, a vertical line $Y' OY$ intersecting at O, the *origin*. We select a convenient unit of length and starting from the origin as zero, mark off a number scale on the horizontal line, positive to the right and negative to the left. We mark off the *same scale* on the vertical line, positive upwards and negative downwards of the origin O. The horizontal line thus marked is called the *x-axis* and the vertical line the *y-axis*, and collectively they are called the *coordinate axes*.

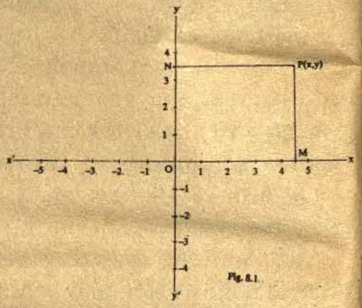


Fig. 8.1

Let P be any point in the plane. Draw perpendiculars from P to the coordinate axes, meeting the x-axis in M and the y-axis in N (Fig. 8.1). Let x be the length of the *directed line segment OM* in the units of the scale chosen. This is called the *x-coordinate* or *abscissa* of P. Similarly, the length of the directed line segment ON in the same scale is called the *y-coordinate* or *ordinate* of P. The position of the point P in the plane with respect to the coordinate axes is represented by the *ordered pair* (x, y) of real numbers, writing the abscissa first in the parenthesis. The pair (x, y) is called the *coordinates* of P, and this system of coordinating an ordered pair (x, y) with every point of the plane is called the (Rectangular) *Cartesian Coordinate System.*

We thus see that to every point P in the Euclidean plane there corresponds a unique ordered pair (x, y) of real numbers called its Cartesian Coordinates. Conversely, given an ordered pair (x, y) and a cartesian coordinate system, we mark off a directed line segment $QM = x$ on the x-axis and another directed line segment $QN = y$ on y-axis, draw perpendiculars at M and N to x and y-axes respectively, and their point of intersection shall uniquely locate the corresponding point P in the Euclidean plane. This establishes a 1-1 correspondance between the set of all ordered pairs (x, y) of real numbers and the points in the Euclidean plane. The set of all ordered pairs (x, y) of real numbers is called the number plane and is denoted by R^2.

Finally we observe (Fig. 8.2) that the two axes divide the plane into four regions called the *quadrants*. The ray OX is taken as positive x-axis, OX' as negative x-axis, OY as positive y-axis and OY' as negative y-axis. The quadrants are thus characterized by the following signatures of abscissa and ordinate.

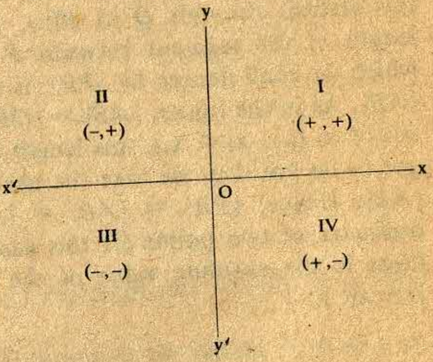


Fig. 8.2

I	quadrant	$x>0$, $y>0$	or (+,+)
II	quadrant	$x<0$, $y>0$	or (−,+)
III	quadrant	$x<0$, $y<0$	or (−,−)
IV	quadrant	$x>0$, $y<0$	or (+,−)

Further if the abscissa of a point is zero, it would lie somewhere on the y-axis and if its ordinate is zero it would lie on x-axis. Thus by simply looking at the coordinates of a point we can tell in which quadrant it would lie, e.g. the points (3, 4), (1,−2), (−2, −3) and (−4, 5) lie respectively in I, IV, III and II quadrants.

8.3 Distance Formula

The distance between any two points in the plane is the length of the line segment joining them. Let the coordinates of these two points P and Q be (x_1, y_1) and (x_2, y_2) respectively. We shall sometimes refer to them as points $P(x_1, y_1)$ and $Q(x_2, y_2)$ and obtain, as under, a formula for the distance between them.

Theorem 8.1

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

where the square root is taken to be positive.

Proof: Let P (x_1, y_1) and Q (x_2, y_2) be the two points in the plane, and let d be the distance between them (Fig. 8.3). Draw lines parallel to y-axis from the points $P(x_1, y_1)$ and Q (x_2, y_2) which will meet the x-axis in points $A(x_1, 0)$ and B $(x_2, 0)$ respectively. Now draw a line through P (x_1, y_1) parallel to x-axis which will meet the vertical through Q in $R(x_2, y_1)$. The length of the segment between P and R, which we shall denote by $|PR|$, is equal to $|AB|$. As in the figure, $|AB| = |OB| - |OA| = (x_2-x_1)$. If, however, x_2 were to the left of x_1 (i.e. $x_2 < x_1$), this length were (x_1-x_2). In other case, since the length has got to be positive, we take the absolute value of (x_2-x_1) viz $|x_2-x_1|$ as the length $|AB|$. Hence, $|PR| = |AB| = |x_2-x_1|$. In passing we observe that when the ordinates of two points (in this case P and R) are the same, the distance between them is the absolute value of the difference between their abscissa (in this case $|x_2-x_1|$).

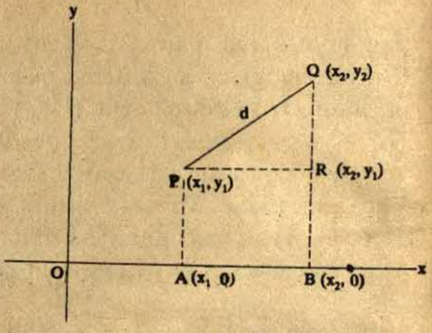


Fig. 8.3

Repeating the same argument for points Q and R, by drawing lines parallel to x axis and meeting the y-axis, we shall find that the length $|RQ| = |y_2-y_1|$. (What do you observe in this case? It would also be a good exercise to check the validity of these facts when P and Q lie in different quandrants).

Now applying the Pythagoras theorem, we get

$$|PQ|^2 = |PR|^2 + |RQ|^2 \text{ or } d^2 = |x_2-x_1|^2 + |y_2-y_1|^2$$

which can also be written as

$$d^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

Since the distance is always positive, taking the positive square root, we get the *distance formula*

$$d = |PQ| = \sqrt{(x_2-x_1)^2 + (y_2 - y_1)^2}$$

Corollary: The distance of any point P (x, y) from the origin is $\sqrt{x^2+y^2}$.

Proof: In the above formula, take the point P as (x, y) and Q as $(0, 0)$, i.e. the origin, to get the result.

Example 8.1

What is the distance between the points (4, 5) and (−3, 2)?

Solution

It is immaterial whether we select (4, 5) or (−3, 2) as P, since the choice affects only the sign of (x_2-x_1) and (y_2-y_1). If we take (4, 5) as Q, then

$$\begin{aligned} PQ &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{[4-(-3)]^2+(5-2)^2} \\ &= \sqrt{(4+3)^2+(3)^2} \\ &= \sqrt{(7)^2+(3)^2} \\ &= \sqrt{58} \end{aligned}$$

Example 8.2

Prove that the points (4, 4), (3, 5) and (−1, −1) represent the vertices of a right triangle.

Proof: Let the points (4, 4), (3, 5) and (−1, −1) represent the points P, Q and R respectively. Then,

$$PQ = \sqrt{(3-4)^2+(5-4)^2} = \sqrt{2}$$

$$QR = \sqrt{[-1-(3)]^2+[-1-(5)]^2} = \sqrt{52}$$

and $$PR = \sqrt{[-1-(4)]^2+[-1-(4)]^2} = \sqrt{50}$$

Since $QR^2 = RP^2+PQ^2$, it follows from the converse of the Pythagoras Theorem that the triangle is a right triangle, with right angle at P.

EXERCISE 8.1

1. Find the distance between each of the following pairs of points:
   (i) (−1, −4), (3, 5)
   (ii) $(a \cos \alpha, a \sin \alpha)$, $(a \cos \beta, a \sin \beta)$
2. By using the distance formula, prove that each of the following sets of points are the vertices of a right triangle:
   (i) (6, 2), (3, −1), (−2, 4)
   (ii) (−2, 2), (8, −2), (−4, −3)
3. Show that each of the triangles whose vertices are given below are isosceles:

(i) (8, 2), (5, −3), (0, 0)
(ii) (0, 6), (−5, 3), (3, 1)

4. Find the value of x such that $PQ = QR$, where P, Q and R are (6, −1), (1,3) and (x, 8), respectively.
5. What point on the x-axis is equidistant from (7, 6) and (−3, 4)?
6. Give the relation that must exist between x and y so that (x, y) is equidistant from (6, −1) and (2, 3).
7. Show that the quadrilateral with vertices (3, 2), (0, 5), (−3, 2), (0, −1) is a square.

8.4 Area of a Triangle

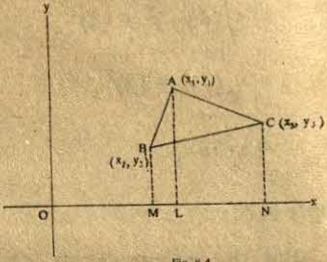


Fig. 8.4

We now proceed to find the area of a triangle ABC, the coordinates of whose vertices are given to be $A\ (x_1, y_1)$, $B\ (x_2, y_2)$ and $C\ (x_3, y_3)$. Draw perpendiculars from A, B and C to x-axis meeting it in L, M and N respectively. As we have seen earlier, $|LM|$ or simply $ML = |x_1 - x_2| = x_1 - x_2$. Similarly, $LN = (x_3 - x_1)$ and $MN = x_3 - x_2$. Now, the area of ΔABC = area of trapezium $BMLA$ + area of trapezium $ALNC$ − area of trapezium $BMNC$ =
$$\frac{1}{2}(MB+AL).\ ML + \frac{1}{2}(AL + CN).\ LN - \frac{1}{2}(BM+CN).MN = \frac{1}{2}(y_2+y_1)(x_1-x_2) +$$
$$\frac{1}{2}(y_1+y_3)(x_3-x_1) - \frac{1}{2}(y_2+y_3)(x_3-x_2) = \frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]$$

This expression for the area can also be expressed in the determinant form as follows:

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Condition for Collinearity of Three Points

Three points $A\ (x_1, y_1)$, $B\ (x_2, y_2)$ and $C\ (x_3, y_3)$ are collinear, i.e. lie on the same straight line if and only if the area of the ΔABC is zero. Hence, three points $A\ (x_1, y_1)$, $B\ (x_2, y_2)$ and $C\ (x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Note: It may turn out sometimes that the numerical value of the determinant giving the area of the ΔABC may be negative. The area however shall always be taken to be positive. If you interchange the order of two vertices of the triangle ABC, the two rows of the determinant are interchanged and it becomes opposite in sign. Conventionally we take the vertices in anti-clockwise direction.

Example 8.3

Find the area of a triangle whose vertices are (4, 4), (3, −2) and (−3, 16).

Solution

Using the above formula, required area is

$$\frac{1}{2}\begin{vmatrix} 4 & 4 & 1 \\ 3 & -2 & 1 \\ -3 & 16 & 1 \end{vmatrix} = -27$$

Rejecting the negative sign,

area of $\Delta = 27$ sq. units.

Note: If we actually plot the vertices and take them in anti-clockwise direction, the order would be (4, 4), (−3, 16) and (3, −2), and then the value of the determinant would be +27

Example 8.4

Show that the three points (−1, −1), (2, 3) and (8, 11) lie on a line.

Proof: We have $\begin{vmatrix} -1 & -1 & 1 \\ 2 & 3 & 1 \\ 8 & 11 & 1 \end{vmatrix} = -1(3-11) + 1(2-8) + 1(22-24) = 0.$

Hence the result.

EXERCISE 8.2

1. Find the area of the triangle with vertices at the points given in each of the problems (a) to (d).
   - (a) (0, 0), (1, 0), (1, 1)
   - (b) (−2, 1), (2, −3), (4, 4)
   - (c) (3, 8), (−4, 2), (5, −1)
   - (d) (2, 7), (3, −1), (−5, 6)
2. Show that each of the following triple of points are collinear:
   - (a) (2, 4), (0, 1), (4, 7)
   - (b) (−2, 5), (2, −3), (0, 1)
   - (c) (−5, 7), (−4, 5), (1, −5)
3. For what value of x will the points $(x, -1)$, (2, 1) and (4, 5) lie on a line?
4. A and B are two points (3, 4) and (5, −2). Find the coordinates of a point P such that $PA = PB$ and the area of triangle $PAB = 10$.

   (*Hint:* We can take the area as 10 or −10, hence two different points.)
5. Find the condition that the point (x, y) may lie on the line joining (3, 4) and (−5, −6).

8.5 Section Formula

Theorem 8.2

Given two points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$. The coordinates of the point P on A which divides the line in the ratio $l{:}m$ (internally) are given by

$$x=\frac{mx_1+lx_2}{l+m},\ y=\frac{my_1+ly_2}{l+m}$$

Proof: Draw lines parallel to y-axis from A, B and P meeting x-axis in C, D and Q respectively. Draw lines parallel to x-axis from A and P meeting PQ and BD in E and R respectively. This being given that $\frac{AP}{PB}=\frac{l}{m}$ · It is easily seen that the two right angled triangles APE and PBR are similar, and hence,

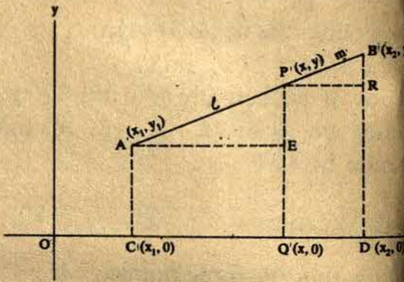


Fig. 8.5

$$\frac{AP}{PB}=\frac{AE}{PR}=\frac{PE}{BR}=\frac{l}{m}, \qquad \text{...(8.1}$$

Now $AE=CQ=|OQ-OC|=|x-x_1|=x-x_1$
and $PR=QD=|OD-OQ|=|x_2-x|=x_2-x$

$$\text{Using}\quad \frac{AP}{PB}=\frac{AE}{PR}=\frac{PE}{BR}=\frac{l}{m}.$$

$$\frac{l}{m}=\frac{AE}{PR}=\frac{x-x_1}{x_2-x}\ \text{or}\ l(x_2-x)=m(x-x_1)$$

$$\text{i.e. } x=\frac{mx_1+lx_2}{l+m} \qquad \text{...(8.2)}$$

Again, $PE=|PQ-QE|=|PQ-AC|=|y-y_1|=y-y_1$

and $BR=|BD-RD|=|BD-PQ|=|y_2-y|=m(y-y_1)$

Using $\frac{l}{m}=\frac{PE}{BR}=\frac{y-y_1}{y_2-y}$, we have $l(y_2-y)=m(y-y_1)$

$$\text{i.e. } y=\frac{my_1+ly_2}{l+m} \qquad \text{...(8.3}$$

Note: To remember the formula it is helpful to note that l is multiplied by the coordinate 'away from it', and similarly is m, and the sum then divided by $l+m$. This is diagramatically shown in Fig. 8.6 as aid to memory.

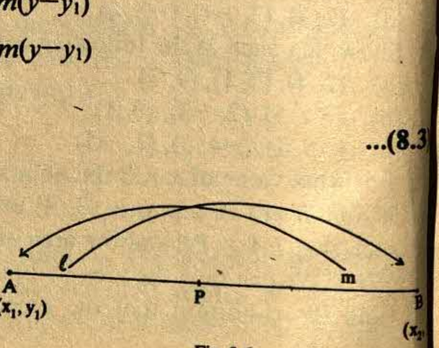


Fig. 8.6

External Division

If the line AB is divided externally in the ratio $l{:}m$, as shown in Fig. 8.7, then to deduce the corresponding section formula we can take help of directed-line-segments AP and PB and

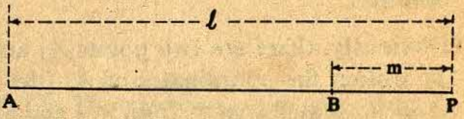


Fig. 8.7

write $$\frac{l}{m} = \frac{AP}{PB} = \frac{AP}{-BP} = \quad \text{or} \quad \frac{|AP|}{|PB|} = \frac{l}{-m}$$

Thus replacing m by $-m$ in the formulas of theorem 8.2 we get the formulas for *external division:*

$$x = \frac{lx_2 - mx_1}{l - m} \qquad y = \frac{ly_2 - my_1}{l - m} \qquad \ldots(8.4)$$

The reader is advised to derive this formula *ab-initio*, proceeding as in Theorem 8.2.

Mid-point Formula

To find the coordinates of the *mid-point* of a line segment with end points $A(x_1, y_1)$ and $B(x_2, y_2)$, we put $l = m$ in the formula of theorem 8.2 and obtain

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}$$

mple 8.5

Find a point on the line through $A(5, -4)$ and $B(-3, 2)$, that is, twice as far from A as from B.

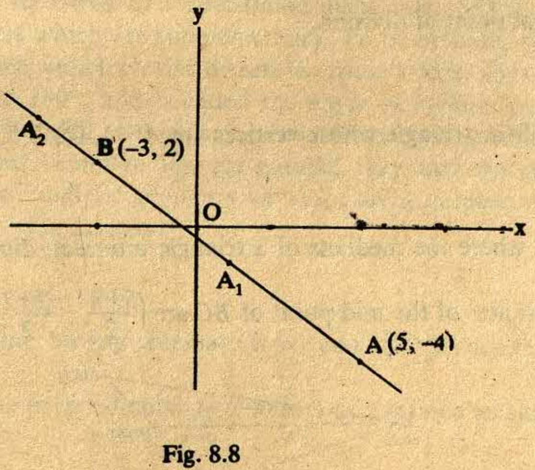


Fig. 8.8

Solution

Obviously, there are two points A_1 and A_2 that satisfy this requirement (See Fig 8. To obtain the coordinates of A_1 (the internal point of division of AB), we may ta A as P_1, B as P_2, $m_1 = 2$, $m_2 = 1$ and apply theorem 8.2. We get

$$x = \frac{1(5)+2(-3)}{2+1} = \frac{5-6}{3} = -\frac{1}{3}$$

and $$y = \frac{1(-4)+2(2)}{2+1} = \frac{0}{3} = 0$$

Thus, the coordinates of A_1 are $(-\frac{1}{3}, 0)$.

To compute the coordinates (x, y) of A_2 (the external point of division of A the ratio must be negative. We make the same choices for P_1 and P_2, but this ti we choose $m_1 = -2$ and $m_2 = 1$. Then

$$x = \frac{1(5)+(-2)(-3)}{-2+1} = -11$$

and $$y = \frac{1(-4)+(-2)(2)}{-2+1} = 8$$

Hence, the coordinates of A_2 are $(-11, 8)$. The student can easily verify that t choice $m_1 = 2$, $m_2 = -1$ or $m_1 = -6$, $m_2 = 3$ gives the same result. The essent fact to be noted is that $\frac{m_1}{m_2} = -2$.

If the ratio $\frac{m_1}{m_2}$ in Theorem 8.2 is a negative number, the point of division w always be an external point of division.

Example 8.6

Find the centroid of the triangle whose vertices are $A(-1, 0)$, $B(5, -2)$ and $C(8, 2)$.

Solution

Centroid, the point where the medians of a triangle intersect, divides each median the ratio 2:1. Coordinates of the mid-point of BC are $\left(\frac{5+8}{2}, \frac{-2+2}{2}\right)$ i.e. $(\frac{13}{2}, 0)$

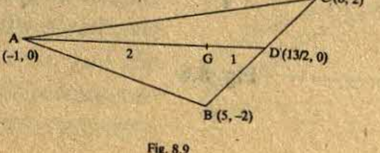


Fig. 8.9

G is the centroid, it must divide the median AD in the ratio 2:1. Hence, by ction formula, its coordinates are

$$= \frac{2\times(13/2)+1(-1)}{2+1} = 4,\ y = \frac{2\times 0+1\times 0}{3} = 0$$

ence, the coordinates of the centroid G are (4, 0). The reader is advised to check e result corresponding to other two medians.

EXERCISE 8-3

. Find the coordinates of the points which divide internally and externally the line joining (1, −3) and (−3, 9) in the ratio 1:3.

. Prove that the points (−2, −1), (1, 0), (4, 3) and (1, 2) are the vertices of a parallelogram.
(*Hint:* Diagonals of a parallelogram bisect each other)

. Find the centroid of a triangle whose vertices are $(x_1 y_1)$, (x_2,y_2) and (x_3,y_3)

. Find the third vertex of a triangle if two of its vertices are at (−1,4) and (5,2) and the centroid at (0, −3).

. In what ratio does the poins $(\frac{1}{2}, 6)$ divide the line segment joining the points (3,5) and (−7, 9)?

. Find the ratio in which the line-segment joining (2,−3) and (5,6) is divided by x-axis.

.6 Slope of a Line

line l not parallel to x-axis in a coordinate plane intersects it. Such a line forms wo angles with x-axis which are supplementary. To be definite, we choose that angle which is made going anticlockwise direction from x-axis. This angle x will have alues between 0 and 180°, and is called the *angle of inclination* or *inclination* of e line l. All lines parallel to x-axis, or coinciding with x-axis, are said to have nclination 0. Note that when two lines are parallel, they have the same inclination.

For the purpose of analytic geometry we associate a number with the inclination f the line in the following manner:

efinition 8.1

he *slope m* of a line having inclination α, and not perpendicular to x-axis, is efined to be tan α.

The slope of a line perpendicular to x-axis is not defined as the value of tan α at $= 90°$ is undefined.

Note that the slope m is independent of the sense of line segment. As shown in Fig. 8.10, slope of $AB = \tan\alpha = \tan(\pi+\alpha)$ = slope of BA and hence we do not take into consideration the direction of a line segment while talking of its slope.

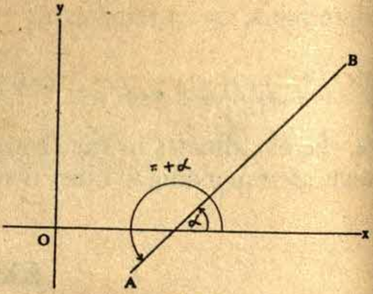


Fig. 8.10

We know that a line is fully determined when we are given two-points on Hence, we proceed to find a formula for the slope of a line in terms of coordinates of two points given on it.

Theorem 8.3

If $P(x_1, y_1)$ and $Q(x_2,y_2)$ are two points on a line l, then the slope m of the lin is given by

$$m = \frac{y_2-y_1}{x_2-x_1}, \quad x_1 \neq x_2$$

If $x_1 = x_2$, then m is not defined. In that case the line is perpendicular to x-axis.

Proof: Let α be the inclination of the line l. We shall consider two different case when α is acute or obtuse, as shown in Fig 8.11(i) and (ii). Draw horizontal an vertical lines (i.e. parallel to x-axis and y-axis respectively) through P and respectively intersecting at M, whose coordinates are shown in the figure.

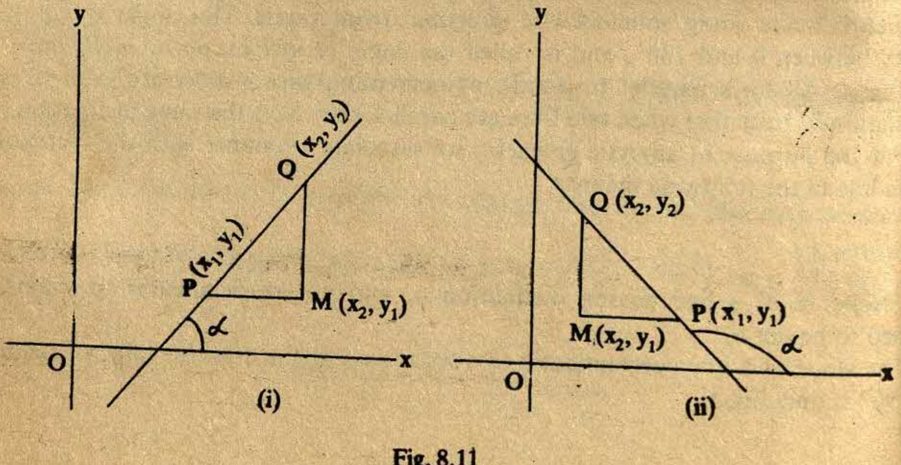


Fig. 8.11

In Fig. 8.11(i) the inclination α is equal to $\angle MPQ$, hence $m = \tan\alpha = \tan\angle MPQ =$

$$\frac{MQ}{PM} = \frac{y_2-y_1}{x_2-x_1}$$

In Fig. 8.11(ii) the inclination α and $\angle MPQ$ are supplementary, hence

$$m = \tan\alpha = \tan(18^\circ - \angle MPQ) = -\tan\angle MPQ = -\frac{MQ}{MP} = -\frac{y_2-y_1}{x_1-x_2} = \frac{y_2-y_1}{x_2-x_1}$$

Consequently, we see that in all cases the slope m of the line through $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$m = \frac{y_2-y_1}{x_2-x_1}$$

The difficulty arises when $x_1 = x_2$, i.e. when m is not defined. This is the case when the line l is parallel to y-axis or perpendicular to x-axis.

Thus given any line not perpendicular to x-axis, we can always find its slope by locating two points on it, and also given any points we can draw a line of the given slope through it as shown in the examples below:

Example 8.7

Find the slope of a line l determined by the points $P(4, 6)$ and $Q(2, 12)$.

Solution.

$$\text{Hence } m = \frac{y_2-y_1}{x_2-x_1} = \frac{12-6}{2-4} = -3$$

Obviously l makes an obtuse angle with x-axis.

Example 8.8

Through $P(4, 1)$ construct a line which has slope equal to $\frac{3}{2}$.

Solution

Starting at P, draw a parallel to x-axis extending to a point R one unit to the right (Fig. 8.12). Now draw a line parallel to y-axis stopping $\frac{3}{2}$ $(= m)$ units above R. The coordinates of this point Q are $(5, \frac{5}{2})$.

Hence, the slope $= \dfrac{2-1}{5-4} = \dfrac{3}{2}$

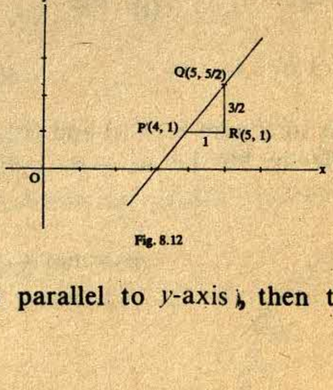


Fig. 8.12

Parallel and Perpendicular Lines

If two lines are parallel (neither of which is parallel to y-axis), then their inclinations are same and hence their slopes are also same.Conversely, if the slope of two lines is the same, then by the property of tangent function there exists unique angle x between 0 and 180° such that $\tan x = m$, and hence their inclinatio is same which means they are parallel. Thus we arrive at the following result.

Theorem 8.4

Two lines (not parallel tc y-axis) are parallel if an only if their slopes are equal.

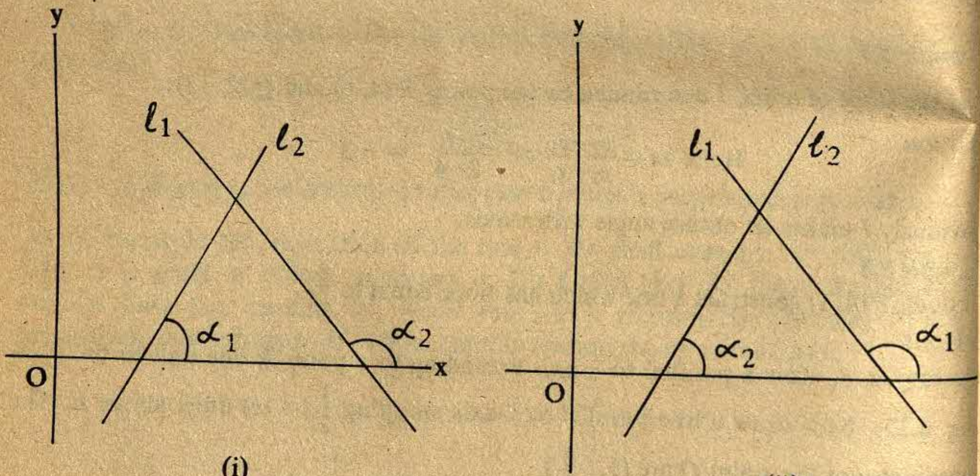
The question arises: Can we characterize perpendicularity also in terms of slop The answer is given by the following theorem:

Theorem 8.5

Two lines (none parallel to y-axis) are perpendicular, if and only if their slopes m m_2 satisfy $m_1m_2 = -1$.

If the slope of one of them is undefined, it is parallel to y-axis and so th perpendicular line has slope zero and is parallel tc x-axis.

Proof



(i) (ii)

Fig. 8.13

As in Figure 8.13 (i) and (ii), if the lines l_1 and l_2 are perpendicular, then α_1 and α_2 differ by 90° i.e. $\alpha_2 = \alpha_1 \pm 90°$, hence $\tan \alpha_2 = \tan(\alpha_1 \pm 90°)$. In either case we get

$$m_2 = \tan \alpha_2 = -\cot \alpha_1 = -\frac{1}{\tan \alpha_1} = -\frac{1}{m_1}$$

$$\text{or } m_1m_2 = -1$$

Conversely, if $m_1 m_2 = -1$ i.e. $\tan \alpha_1 \tan \alpha_2 = -1$, then $\tan \alpha_2 = -\cot \alpha_1$ $= \tan (90° + \alpha)$ or $\tan (\alpha_1 - 90°)$ which is satisfied when α_1 and α_2 differ by 90° i.e. l_1 and l_2 are perpendicular.

Example 8.9

Show that the line through (0, 0) and(2, 3) is parallel to the line through (2, −2) and 6, 4).

Proof:

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-0}{2-0} = \frac{3}{2}$$

$$m_2 = \frac{4-(-2)}{6-2} = \frac{6}{4} = \frac{3}{2}$$

Since the slopes are equal, the lines are parallel.

Example 8.10

Prove that the line through (−2, 6) and (4, 8) is perpendicular to the line through (8, 12) and (4, 24).

Proof:

$$m_1 = \frac{8-6}{4-(-2)} = \frac{1}{3},\ m_2 = \frac{24-12}{4-8} = -3$$

Hence, $m_1 m_2 = -1$ and the lines are perpendicular.

EXERCISE 8.4

1. What can be said regarding a line if its slope is
   (a) positive, (b) zero, (c) negative?
2. What is the slope of a line whose inclination is
   (a) 0°, (b) 45° (c) 90° (d) 150°
3. Find the slope of the line through the points
   (a) (1,2), (4,2) (b) (0,−4), (−6,2) (c) (4,−6), (−2, −5)
4. Show that the line joining (2,−3) and (−5,1) is
   (a) parallel to the line joining (7,−1) and (0,3)
   (b) perpendicular to the line joining (4,5) and (0,−2)
5. State whether the two lines in each of the problems are parallel, perpendicular or neither.
   (a) Through (5,6) and (2,3); through (9,−2) and (6,−5)
   (b) Through (8,2) and (−5,3); through (16, 6) and (3, 15)

Library

(c) Through $(2,-5)$ and $(-2,5)$; through $(6,3)$ and $(1,1)$
(d) Through $(9,5)$ and $(-1,1)$; through $(8,-3)$ and $(3,-5)$

6. Determine x so that 2 is the slope of the line through $(2, 5)$ and $(x, 3)$.
7. What is the value of y so that the line through $(3, y)$ and $(2, 7)$ is parallel to the line through $(-1, 4)$ and $(0, 6)$?
8. Without using Pythagoras Theorem, show that $(4, 4)$, $(3, 5)$ and $(-1, -1)$ are the vertices of a right triangle.
9. A quadrilateral has the vertices at the points $(-4, 2)$, $(2, 6)$, $(8, 5)$ and $(9, -7)$. Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram.

8.7 Locus and Equations

Let us consider the following situations.

Consider a circle of radius a whose centre is at the origin. The circle is the set of points whose distance from the origin is a.

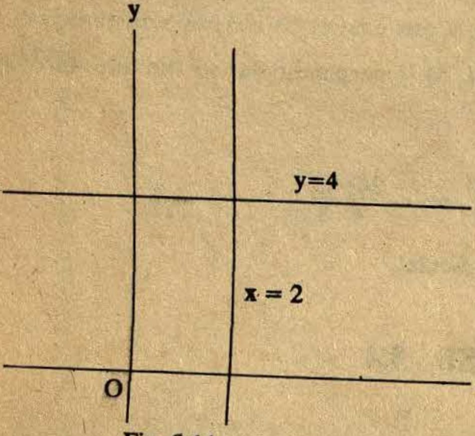


Fig. 8.14

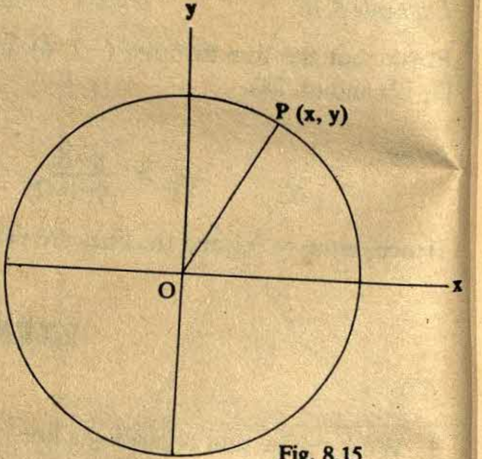


Fig. 8.15

Let $P(x, y)$ be any point on the circle. By the distance formula, the distance from P to O is $\sqrt{x^2+y^2}$, so that we always have $a^2 = x^2 + y^2$. We have thus shown that, if a point lies on a circle of radius a, it must satisfy the equation $x^2+y^2 = a^2$.

Consider a line in the plane parallel to the x-axis and lying 4 units above it. Every point on this line has the property that its ordinate is 4. The equation $y = 4$, therefore characterizes the line. Similarly, a line parallel to y-axis and 2 units away towards its right side is characterized by the equation $x = 2$.

Hence if a point moves according to some fixed rule, its coordinates will always satisfy some corresponding algebraic relation. And the path (curve) of the moving point is called the locus of the point. The curve must contain all the points satisfying the given condition so that no point outside the curve satisfies the condition. Hence, we define the locus in the following way:

Definition 8.2

A locus is the set of all points that satisfy a given geometric condition.

Thus if a point moves so as always to be at a distance a from the origin O, its co-ordinates will always satisfy the equation

$$x^2 + y^2 = a^2$$

Corresponding, therefore, to a curve traced by a point moving according to a fixed rule there exists an algebraic relation. Thus we have an invariable algebraic relation satisfied by the coordinates of every point on the curve, called the *equation of the curve* and conversly, to every algebraic equation, connecting the coordinates of a moving point we have a curve on which the point must lie so long as its coordinates satisfy the given equation.

Thus in the example given above, the equation of the circle whose centre is O. and radius is a is $x^2+y^2 = a^2$ and the locus of the equation $x^2+y^2 = a^2$ is a circle whose centre is O and radius a.

Example 8.11

Find the equation of the locus of points twice as far from (3, 2) as from (1, 1)

Solution

Fig. 8.16 is sketched with the sample point $P(x, y)$.

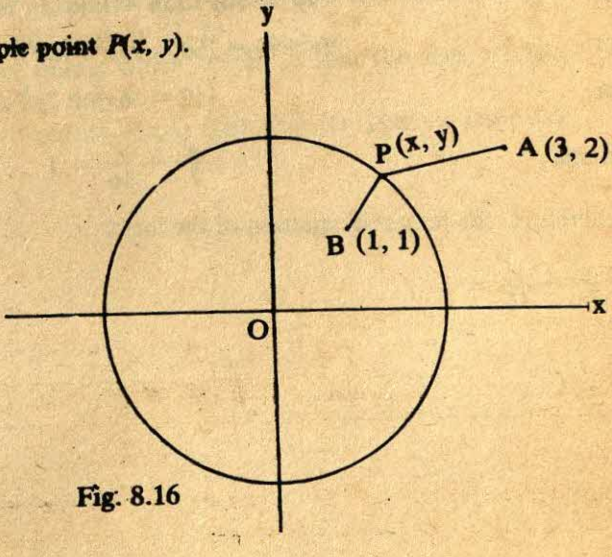


Fig. 8.16

We want $2|PB| = |PA|$. Since $|PB| \geq 0$, and $|PA| \geq 0$ for all P, hence $2\,|PB| = |PA|$, if and only if, $4(PB)^2 = (PA)^2$

This gives $4[(x-1)^2 + (y-1)^2] = (x-3)^2 + (y-2)^2$
or $\quad 4x^2-8x+4+4y^2-8y+4 = x^2-6x+9+y^2-4y+4$
or, $\quad 3x^2 + 3y^2 - 2x - 4y = 5$
Therefore, the locus is given by the equation $3x^2 + 3y^2 - 2x-4y = 5$.

Example 8.12

Find the equation of the locus of points such that the sum of its distances from (0,3) and (0,−3) is 8.

Solution

Let $P(x,y)$ be any point of the locus and A and A' be the points (0,3) and (0,−3) respectively (See Fig. 8.17. We are given that $|PA| + |PA| = 8$. Using this, we have

$$\sqrt{(x-0)^2 + (y-3)^2} + \sqrt{(x-0)^2 + (y+3)^2} = 8$$

or $$x^2 + (y^2-6y+9) = 64-16\sqrt{x^2+(y+3)^2} + x^2+y^2+6y+9$$

or $$-12y = 64-16\sqrt{x^2+(y+3)^2}$$

or $$12y + 64 = 16\sqrt{x^2+(y+3)^2}$$

or $$3y + 16 = 4\sqrt{x^2+(y+3)^3}$$

or $$9y^2 + 96y+256 = 16(x^2+y^2+6y+9)$$

or $$9y^2+96y+256 = 16x^2+16y^2+96y+144$$

or $$112 = 16x^2 + 7y^2$$

Hence, $$\frac{x^2}{7} + \frac{y^2}{16} = 1$$

Therefore, the required equation of the locus is $\frac{x^2}{7} + \frac{y^2}{16} = 1$

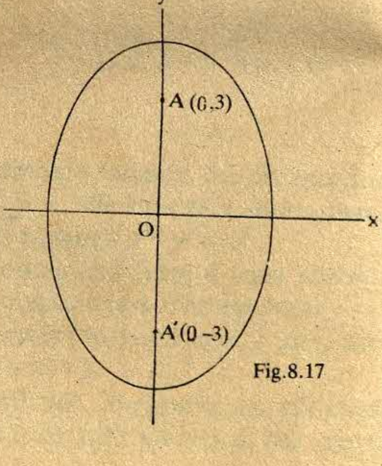


Fig.8.17

rom the above discussion it is clear that if a coordinate system is defined, the ondition determining a particular locus may be expressible as an equation or a set f equations involving the coordinates x and y of a point. When it is possible, as is ften the case, a relationship is established between loci and equations. This relation- hip is the basis of two fundamental problems in coordinate geometry.

1. Give a locus (geometric condition), to find the corresponding algebraic relation (equation).
2. Given an algebraic relation (equation), to find the corresponding locus.

In the articles that follow, we shall consider these two problems with respect to the loci—the straight line and circle.

EXERCISE 8.5

1. Find the equation of the locus ot points equidistant from $(-1, -1)$ and $(4,2)$.
2. Find the equation of the locus of all points equidistant from the point $(4, 2)$ and the x-axis.
3. Find the equation of the locus of all points $P(x,y)$ such that the segment OP has slope 3.
4. Show that the lines whose equations are $x-y=5$ and $x+y=7$ intersect in the point $(6, 1)$.
5. Find the equation of the locus of which every ordinate is greater than the corresponding abscissa by a given distance.
6. Find the equation of the locus of a point such that the sum of its distances from $(0,2)$ and $(0,-2)$ is 6.
7. Find the equation of the locus of all points $P(x,y)$ such that the segment OP is parallel to the segment joining P and the point $(3, 2)$.
8. Find the equation of the locus of points which moves so that the sum of their distances from $(3,0)$ and $(-3, 0)$ is less than 9.
9. Find the equation of the locus of points $P(x,y)$ such that its distance from the point $(-1, 2)$ is greater than 3.

CHAPTER 9

STRAIGHT LINE

9.1 To find the Equation of a Straight Line Parallel to an Axis

We know that on x-axis, the y-coordinates of all points is zero. We say that th equation of x-axis is $y = 0$. Similarly, the equation of y-axis is $x = 0$. Now th equation of a straight line parallel to x-axis is $y = b$, where b is the ordinate of an point on the line. Similarly, $x = a$ is the equation of a straight line parallel to y-axis where a is the abscissa of any point on the line.

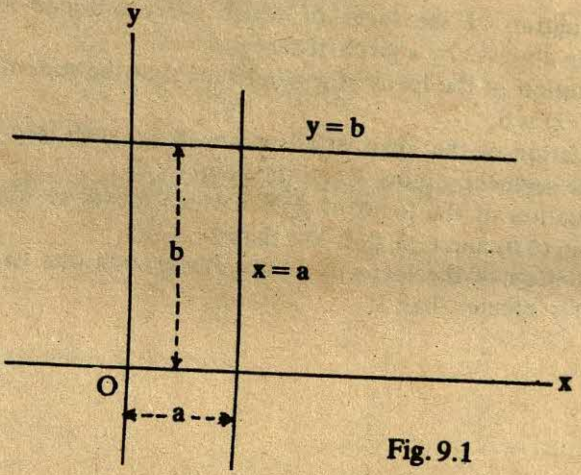


Fig. 9.1

9.2 The Point-Slope Form

Now we are in a position to obtain the equation of a line determined by a given se of conditions.

Let $P_1\ (x_1, y_1)$ be a fixed point and m be a given slope. If any other point on the line is $P\ (x,y)$, then $\frac{y-y_1}{x-x_1}$ is the slope of the line through P_1 and P. But this is given as m. Hence,

$$\frac{y-y_1}{x-x_1}=m \qquad \text{...(9.1)}$$

or $y-y_1=m(x-x_1)$

Equation (9.1) is the equation to the line having slope m and passing through the point P_1 (x_1, y_1). It is known as the point-slope form of the equation of a straight line, since it is in terms of a point and the slope.

Note that the slope m is undefined for the lines parallel to y-axis. Hence, the point-slope form of the equation will not be applicable to the equation of a line through P_1 (x_1, y_1) parallel to the y-axis. However, this presents no difficulty, since for any such line the x-coordinate of any point on it is x_1, the equation of such a line is $x=x_1$.

Example 9.1

Determine the equation of the line passing through the point $(-1, -2)$ and with slope $\frac{4}{7}$.

Solution

Putting the values $x_1=-1$, $y_1=-2$ and $m=\frac{4}{7}$ in the point-slope form of the equation, we get

$$y-(-2)=\frac{4}{7}[x-(-1)]$$

or

$$y+2=\frac{4}{7}(x+1)$$

or

$$7y+14=4x+4$$

Therefore, equation of the line is $y=\frac{4}{7}x-\frac{10}{7}$.

Example 9.2

Determine the equation of the line through the point $(3, -4)$ and parallel to the x-axis.

Solution

A line parallel to the x-axis has the slope zero. Therefore, point-slope form of the equation gives

$$y-(-4)=0\,(x-3)$$

or

$$y+4=0$$

Another way to approach this problem is to note that every point on the line must have the same ordinate. Since one point has ordinate -4, we must have $y=-4$ for all points.

9.3 Two-Point Form

Let Q_1 (x_1, y_1) and Q_2 (x_2, y_2) be two given points and P (x, y) be any point on t line.

Draw the ordinates Q_1N_1, Q_2N_2, PM and draw Q_1RS parallel to OX to cut PM in and Q_2N_2 in S. Then $Q_1R = OM - ON_1 = x - x_1$

$$PR = PM - RM = PM - Q_1N_1 = y - y_1$$
$$Q_1S = ON_2 - ON_1 = x_2 - x_1$$
$$Q_2S = Q_2N_2 - SN_2 = Q_2N_2 - Q_1N_1 = y_2 - y_1$$

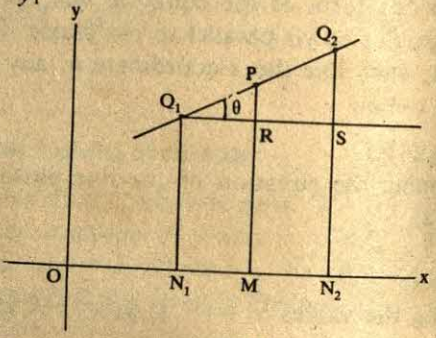


Fig. 9.2

In ΔPQ_1R, $\tan\theta = \dfrac{PR}{Q_1R} = \dfrac{y-y_1}{x-x_1}$ ...(

In ΔQ_2Q_1S, $\tan\theta = \dfrac{Q_2S}{Q_1S} = \dfrac{y_2-y_1}{x_2-x_1}$ ...(ii

From (i) and (ii) we get

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

or $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1}$

which is the equation of the line in the two-point form.

The above equation can also be derived as follows:

Slope of the line $Q_1Q_2 = \dfrac{y_2-y_1}{x_2-x_1}$

By point-slope form, the equation of the line through Q_1 is

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$$

or $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1}$

9.4 Intercept Form

Let l be a straight line. There are two distinguished points on it, namely the two points of intersection of l with x and y-axes. The points where it meets the x-axis is obtained by setting $y = 0$and solving for x. Similarly, the points where it meets the y-axis is obtained by setting $x = 0$ and solving for y. If these points are $(a, 0)$ and $(0, b)$, the numbers a and b are called the *x-intercept* and *y-intercept* respectively. For example, the x and y intercepts for the line $2y = 3x + 5$ are $-\frac{5}{3}$ and $\frac{5}{2}$ respectively. If the line passes through the origin, the x and y intercepts are zero. If the line is parallel to x-axis, its x-intercept is undefined. Similarly, for a line parallel to the y-axis, the y-intercept is undefined.

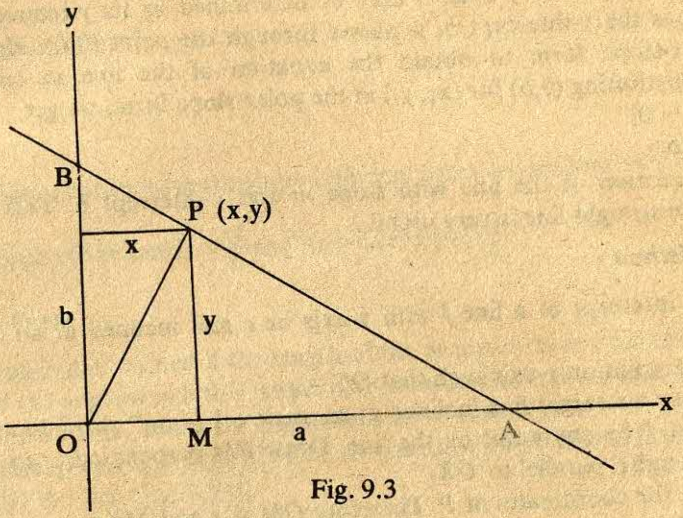


Fig. 9.3

Let the line meet the x-axis in point A and y-axis in point B. So the intercepts on the axes of x and y are $OA\ (= a)$ and $OB\ (= b)$.

Let $P\ (x,y)$ by any point on the line. The geometrical condition satisfied by P is Area of ΔOBP + Area of ΔOAP = Area of ΔOAB

i.e. $\frac{1}{2}bx + \frac{1}{2}ay = \frac{1}{2}ab$

or $\frac{x}{a} + \frac{y}{b} = 1$

which is the equation of the line in the intercept form. Alternatively, this is a particular case of section 9.3 in the following way.

The coordinates of A and B are $(a,0)$ and $(0,b)$ respectively. Using two-point form, the equation of the line is

$$\frac{x-a}{0-a} = \frac{y-0}{b-0}$$

or $-\frac{x}{a} + 1 = \frac{y}{b}$

or $\frac{x}{a} + \frac{y}{b} = 1$ ...(9.3)

which is the required equation.

9.5 Slope-Intercept Form

If a line is not parallel to y-axis, it may be determined by its y-intercept and slope m. If a line has the y-intercept b, it passes through the point $(0,b)$. Hence, we may use the point-slope form to obtain the equation of the line in terms of these quantities. Substituting $(0,b)$ for (x_1, y_1) in the point-slope form, we get

$y - b = m(x - 0)$

or $y = mx + b$

This is the equation of the line with slope m and y-intercept b. This form of the equation of the straight line is very useful.

Alternative Method

Let the given intercept of a line l with y-axis be c and inclined at an angle α with x-axis.

Let C be the point on y-axis such that $OC = c$.

Through C draw a straight line inclined at an angle α [$=\tan^{-1}(m)$] to x-axis so that $\tan\alpha = m$. Let P be any point on the line. Draw PM perpendicular to OX to meet in N a line through C parallel to OX.

Let (x,y) be the coordinates of P. Therefore $OM = x$ and $MP = y$

$\angle PL\,M = \angle PCN$

i.e. $\angle PCN = \alpha$

$MP = NP + MN$

or $NP = MP - NM = MP - OC$

or $NP = y - c$ and $CN = OM = x$

In ΔPCN,

$$\tan\alpha = \frac{PN}{CN}$$

$$\text{or } m = \frac{y-c}{x}$$

or $y = mx + c$

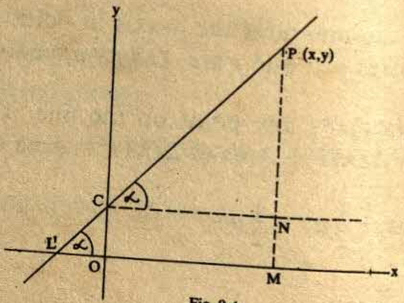


Fig. 9.4

This is the equation of the line with slope m and y-intercept c. This form of the equation of the straight line is very useful. ...(9.4)

Example 9.3

What is the equation of a line with slope 3 and y-intercept 2?

Solution

On substituting $m = 3$ and $b = 2$, in the slope intercept form of the equation, we get

$y = 3x + 2$

This is the desired equation.

Example 9.4

Determine the slope and the y-intercept of the line whose equation is $5x + 6y = 7$.

Solution

Solving for y, we have

$$y=\frac{5}{6}x+\frac{7}{6}$$

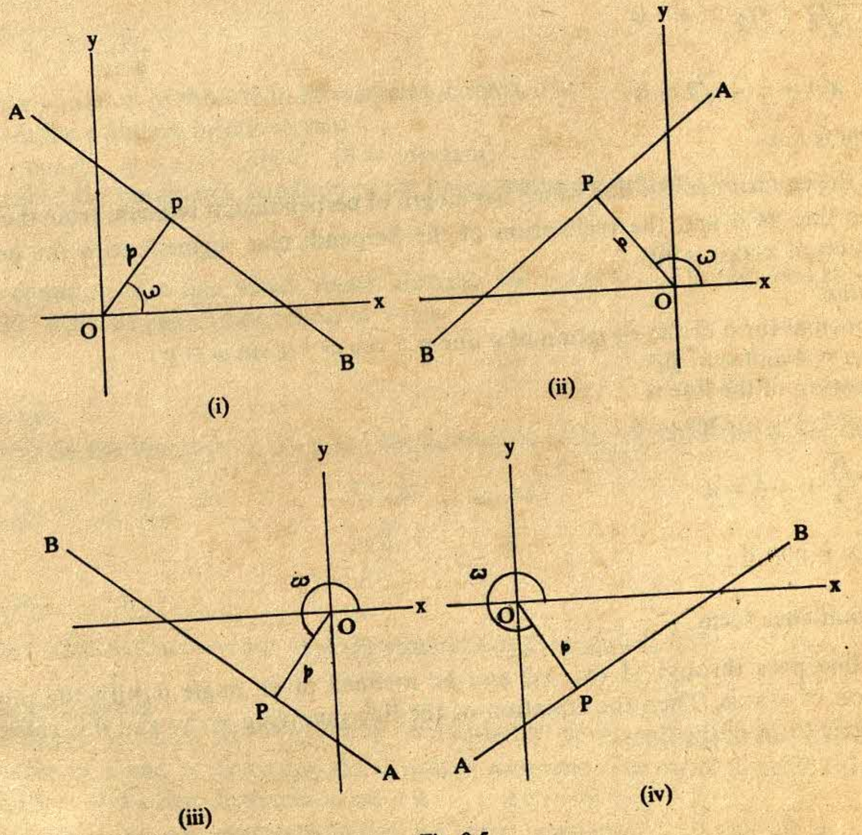
Comparing this equation with the slope-intercept form, we see that $m=-\frac{5}{6}$ and $b=\frac{7}{6}$.

Therefore, slope of the line is $-\frac{5}{6}$ and its y-intercept is $\frac{7}{6}$.

9.6 Normal Form

A straight line is determined if the length of the perpendicular from the origin (0,0) to the line, and the angle which this perpendicular makes with the x-axis are known.

Let AB be the line. Draw OP perpendicular to AB as shown in figures (9.5). Four different figures **[See Fig.9.5 (i),** (ii), (iii), (iv)] are given for various positions of the line AB. Consider the first Fig. 9.5 (i).



(i) (ii) (iii) (iv)

Fig. 9.5

Let ω be the angle between OP and the positive x-axis, and p the length of the perpendicular OP. Then the coordinates of the point p will be $(P\cos\omega,\ p\sin\omega)$, and the slope of AB will be $-\dfrac{1}{\tan\omega} = -\cot\omega$. If (x,y) is any other point on the line AB, then by the point-slope formula, $y - p\sin\omega = -\cot\omega\,(x - p\cos\omega)$. On simplifying, we get

$$x\cos\omega + y\sin\omega - p = 0$$

$$\text{or } x\cos\omega + y\sin\omega = p \qquad \text{...(9.5)}$$

which is the perpendicular form of the straight line. It can be verified that we get the same form, if we consider Fig. (i) (ii) (iii) and (iv).

Example 9.5

Find the equation of the line with $\omega = 135°$ and perpendicular distance 4.

Solution

From (9.5), we have

$$x\cos 135° + y\sin 135° - 4 = 0$$

$$\text{or } -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - 4 = 0$$

$$\text{or } -x + y - 4\sqrt{2} = 0$$

Example 9.6

Find the equation of the line which has length of perpendicular segment from the origin to the line as 4 and the inclination of the perpendicular segment with the positive direction of x-axis is 30°.

Solution

The normal form of the equation of a line is $x\cos\omega + y\sin\omega = p$

Here $p = 4$ and $\omega = 30°$

∴ Equation of the line is

$$x\cos 30° + y\sin 30° = 4$$

$$\text{or } x\frac{\sqrt{3}}{2} + y\frac{1}{2} = 4$$

$$\text{or } \sqrt{3}x + y = 8$$

9.7 Symmetric form

Let a line pass through $A(x_1, y_1)$ and be inclined at an angle θ with the positive direction of x-axis. Then the equation of the line involving x_1, y_1 and θ is called the symmetric form of the line.

Let $P(x, y)$ be any point and $AP = r$. Draw AB and PC perpendiculars to x-axis from A and P respectively and $AN \perp PC$.
Now, $AN = BC = OC - OB = x - x_1$
$PN = PC - CN = PC - AB = y - y_1$
Also $AP = r$

In ΔANP, $\cos\theta = \dfrac{AN}{AP}$
$= \dfrac{(x - x_1)}{r}$

i.e., $\dfrac{x - x_1}{\cos\theta} = r$ ...(i)

and $\sin\theta = \dfrac{PN}{AP} = \dfrac{(y - y_1)}{r}$

i.e., $\dfrac{y - y_1}{\sin\theta} = r$ ...(ii)

From (i) and (ii)

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} \quad ...(9.6)$$

which is the equation of the line in the symmetric form.

Note: From the equation it follows that
$x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$, (θ = constant)
which is called the parametric equation of the line, r being the parameter.

Example 9.7

Find the equation of a line which passes through the point $(-2, 3)$ and makes an angle of 30° with the positive direction of x-axis.

Solution

Here $\theta = 30°$
Given point on the line is $(-2, 3)$. Using the symmetric form, the equation of the line is

$$\frac{x+2}{\cos 30°} = \frac{y-3}{\sin 30°} \quad \text{or} \quad \frac{x+2}{\frac{\sqrt{3}}{2}} = \frac{y-3}{\frac{1}{2}} \quad ...(9.7)$$

or $\sqrt{3}y - 3\sqrt{3} = x + 2$
or $x - \sqrt{3}y + (3\sqrt{3} + 2) = 0$ is the equation of the required line.

9.8 General Form

All the forms in which we have found the equation of the straight line are of the first degree in x and y. The converse of this is also true. The most general form of any equation of the first degree in x and y is $Ax + By + C = 0$

This equation, though apparently involving three constants A, B and C, in reality involves only two— namely,the ratios $\frac{A}{C}$ and $\frac{B}{C}$.

We shall now show that every straight line is represented by an equation of the first degree and conversely.

Theorem 9.1

The equation $Ax + By + C = 0$ always represents a straight line provided A and B *are not zero simultaneously*.

Proof: We shall now consider the following two cases:

Case 1 : If $B = 0$ and $A \neq 0$, then the equation $Ax + By + C = 0$ becomes $Ax + C = 0$ or $x = -\frac{C}{A}$ and is satisfied by all points lying on a line parallel to y-axis and at a distance $-\frac{C}{A}$ units from it.Hence,this is the equation of a straight line. The case where $B \neq 0$, and $A = 0$, can be treated similarly.

Case II: If $B \neq 0$ and $A \neq 0$, we can solve the equation for y and obtain

$$y = -\frac{A}{B}x - \frac{C}{B}$$

This represents the straight line with slope $-\frac{A}{B}$ and y-intercept $-\frac{C}{B}$.

The converse is given in the following theorem:

Theorem 9.2

Every straight line has an equation of the form $Ax + By + C = 0$, where A, B and C are constants.

Proof : Given a straight line, either it cuts the y-axis, or is parallel to or coincident with it. We know that the equation of a line which cuts the y-axis(that is it has a y-intercept) can be put in the form $y = mx + b$, further, if the line is parallel or coincident with the y-axis,its equation is of the form $x = x_1$, where $x_1 = 0$ in the case of coincidence. Both of these equations are of the form given in the theorem; hence the proof.

We can use the form $Ax + By + C = 0$ to determine the equation of a straight line in the following way:

Example 9.8

Find the equation of the line through (3, 4) and (2, −1).

Solution

We seek the numbers A, B and C such that the line $Ax + By + C = 0$ passes through the two given points. If $Ax + By + C = 0$ passes through (3, 4), then $3A + 4B + C = 0$; if it passes through (2, −1), then $2A - B + C = 0$. If (x, y) is any point on the line,then

$$\left.\begin{aligned} Ax + By + C &= 0 \\ \text{Also } 3A + 4B + C &= 0 \\ 2A - B + C &= 0 \end{aligned}\right\} \quad \ldots(i)$$

Since these equations have non-trivial solution, eliminating A, B and C from (i), we have

$$\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 0$$

Expanding the determinant, we have

$$y = 5x - 11$$

Therefore, the equation of the line through (3, 4) and (2, −1) is $y = 5x - 11$.

The general equation of straight line is reducible to the normal form in the following way:

The general equation of the straight line is

$$Ax + By + C = 0 \qquad ...(9.8)$$

The equation of a line in the normal form is

$$x \cos \alpha + y \sin \alpha - p = 0 \qquad ...(9.9)$$

If we suppose that (9.8) and (9.9) represent the same straight line, then we can compare the corresponding coefficients

$$\frac{A}{\cos \alpha} = \frac{B}{\sin \alpha} = \frac{C}{-p}$$

or $\cos \alpha = -\dfrac{Ap}{C}$ and $\sin \alpha = -\dfrac{Bp}{C}$

Squaring and adding both, we have

$$\cos^2\alpha + \sin^2\alpha = \frac{A^2p^2}{C^2} + \frac{B^2p^2}{C^2}$$

or $1 = \dfrac{p^2}{C^2}(A^2 + B^2)$

or $p^2 = \dfrac{C^2}{A^2+B^2}$

or $p = \pm \dfrac{C}{\sqrt{A^2+B^2}}$

But p is the measure of the perpendicular segment, and is always positive.

Hence, $p = \dfrac{C}{\sqrt{A^2+B^2}}$

Therefore, $\cos \alpha = -\dfrac{A}{\sqrt{A^2+B^2}}$

and $\sin \alpha = -\dfrac{B}{\sqrt{A^2+B^2}}$

Hence, (9.9) takes the form

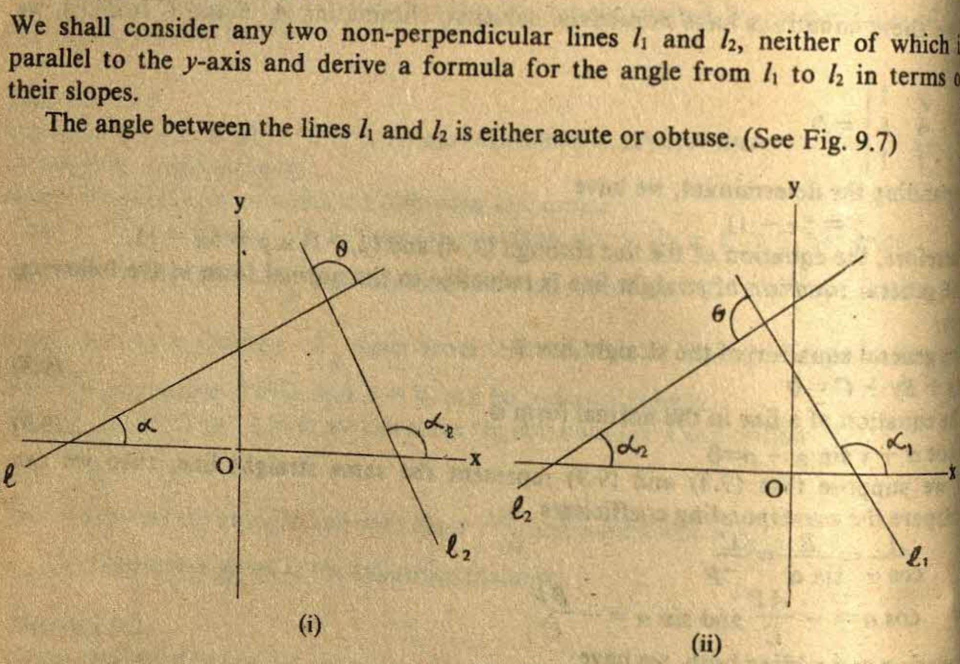
$$-\frac{A}{\sqrt{A^2+B^2}}\,x - \frac{B}{\sqrt{A^2+B^2}}\,y = \frac{C}{\sqrt{A^2+B^2}}$$

which is the required equation in normal form.

9.9 Angle between Two Lines

We shall consider any two non-perpendicular lines l_1 and l_2, neither of which i parallel to the y-axis and derive a formula for the angle from l_1 to l_2 in terms o their slopes.

The angle between the lines l_1 and l_2 is either acute or obtuse. (See Fig. 9.7)



(i)

(ii)

Fig. 9.7

From Fig. 9.7 (i), we see that

$$\alpha_2 = \alpha_1 + \theta,$$

since α_2 is an exterior angle of a traingle with α_1 and θ as the opposite interior angles. Therefore,

$$\theta = \alpha_2 - \alpha_1$$

and $\tan\theta = \tan(\alpha_2 - \alpha_1)$

$$= \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2}$$

Therefore, $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$, ...(9.10)

where $m_1 = \tan\alpha_1$, and $m_2 = \tan\alpha_2$

From Fig. 9.7 (ii), we see that

$\alpha_1 = \alpha_2 + (\pi - \theta)$,

since $(\pi - \theta)$ and α_2 are interior angles with α_1 as the opposite exterior angle.

Therefore, $\theta = \alpha_2 - \alpha_1 + \pi$

or $\tan\theta = \tan(\pi + (\alpha_2 - \alpha_1))$

$= \tan(\alpha_2 - \alpha_1)$

$$= \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_2 \tan \alpha_1}$$

us $\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$.

e have thus proved the following theorem:

ieorem 9.3

ie positive angle θ from the line l_1 to the line l_2 with slopes $m_1 = \tan \alpha_1$ and $_2 = \tan \alpha_2$ respectively is given by equation

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

We note that if the two lines are perpendicular to each other, then, as seen earlier,

$$m_1 = -\frac{1}{m_2}, \text{ that is } 1 + m_1 m_2 = 0.$$

ius $\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$ is not defined in this case.

Notice that in numerical examples, the value of $\tan \theta$ will sometimes be found to e negative. This would merely mean that instead of getting the acute angle of tersection, its supplement, which too is the angle of intersection of the lines, is eing obtained.

xample 9.9

etermine the angle B of the triangle with vertices $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$.

olution

et BA be l_1, BC be l_2 so that formula (9.10) will give the desired angle (See ig. 9.8)

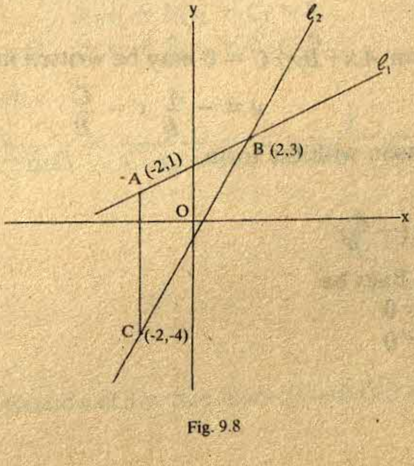


Fig. 9.8

$$\text{Then, } m_1 = \frac{1-3}{-2-2} = \frac{1}{2}$$

$$m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

$$\text{and } \tan \angle B = \frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \cdot \frac{1}{2}} = \frac{\frac{5}{4}}{\frac{15}{8}} = \frac{2}{3} = 0.667$$

Thus $\angle B \cong 33°42$

Example 9.10

Find the value of m_1 if $m_2 = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$.

Solution

From the formula

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2},$$

we get
$$\tan \frac{\pi}{4} = \frac{\frac{1}{2} - m_1}{1 + \frac{m_1}{2}}$$

or
$$1 = \frac{1 - 2m_1}{2 + m_1}$$

or
$$2 + m_1 = 1 - 2m_1$$

Therefore, $m_1 = -\frac{1}{3}$

The equation $Ax + By + C = 0$ may be written in the form

$$y = -\frac{A}{B}x - \frac{C}{B}$$

from which, by comparison with the form

$$y = mx + b,$$

we see that the slope is $(-\frac{A}{B})$

Let the equation of the lines be

$$A_1x + B_1y + C_1 = 0$$

and
$$A_2x + B_2y + C_2 = 0$$

In the slope form

$$y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1}$$

and $y = -\frac{A_2}{B_2}x - \frac{C_2}{B_2}$ ...(iv)

If m_1 and m_2 are slopes of (iii) and (iv), then

$m_1 = \frac{-A_1}{B_1}$ and $m_2 = \frac{-A_2}{B_2}$.

We know that the condition for perpendicularity of two lines whose slopes are m_1 and m_2 is

$m_1 m_2 = -1$

This means $\left(\frac{-A_1}{B_1}\right)\left(\frac{-A_2}{B_2}\right) = -1$

or $\frac{A_1A_2}{B_1B_2} = -1$

or $A_1A_2 + B_1B_2 = 0$ ...(9.11)

which is the condition for perpendicularity of the lines given in the general form.

.10 Condition for Concurrency of Three Straight Lines

Let $A_1x + B_1y + C_1 = 0$ ... (i)

and $A_2x + B_2y + C_2 = 0$ ... (ii)

e the two straight lines AL_1 and AL_2 respectively intersecting each other at the oint A as shown in Fig. 9.9.

Since (i) is the equation of AL_1, the coordinates of any point on it must satisfy he equation (i). Similarly, the coordinates of any point on AL_2 must satisfy (ii).

Now, the only point which is common to both the straight lines is their point of ntersection A. The coordinates of this point must, therefore satisfy both (i) and (ii). f (x_1, y_1) are the coordinates of A, then we have,

$A_1x_1 + B_1y_1 + C_1 = 0$ ...(iii)

and $A_2x_1 + B_2y_1 + C_2 = 0$ ...(iv)

olving (iii) and (iv), we have

$$\frac{x_1}{B_1C_2 - B_2C_1} = \frac{y_1}{C_1A_2 - A_1C_2} = \frac{1}{A_1B_2 - B_1A_2}$$

This means

$$x_1 = \frac{B_1C_2 - B_2C_1}{A_1B_2 - B_1A_2}$$

nd $y_1 = \frac{C_1A_2 - A_1C_2}{A_1B_2 - B_1A_2}$

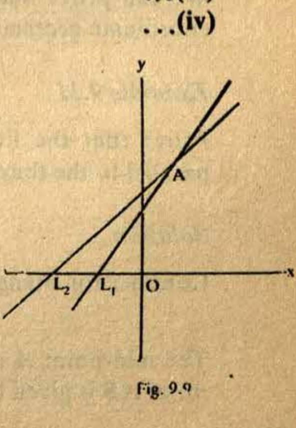


Fig. 9.9

Hence, the point of intersection of the two lines (i) and (iii) are

$$\left(\frac{B_1C_2-B_2C_1}{A_1B_2-B_1A_2}, \frac{C_1A_2-A_1C_2}{A_1B_2-B_1A_2}\right).$$

Now the condition that three lines whose equations are

$$A_1x+B_1y+C_1=0 \qquad \ldots$$
$$A_2x+B_2y+C_2=0 \qquad \ldots($$
$$A_3x+B_3y+C_3=0 \qquad \ldots($$

should be concurrent is that the point of intersection of (v) and (vi) must lie on (vi In other words, the coordinates of the point of intersection of (v) and (vi) shou satisfy the equation of (vii) i.e.

$$A_3\left(\frac{B_1C_2-B_2C_1}{A_1B_2-A_2B_1}\right)+B_3\left(\frac{C_1A_2-C_2A_1}{A_1B_2-A_2B_1}\right)+C_3=0$$

or $$A_3(B_1C_2-B_2C_1)+B_3(C_1A_2-C_2A_1)+C_3(A_1B_2-A_2B_1)=0$$

which is the required condition for concurrency.

Notice that the condition of concurrency requires that the equations (v), (vi) ar (vii) should have a common solution in x and y. Now eliminating x and y from t equations, the condition for this is

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=0 \qquad \ldots(9.1$$

The above equation is simply the expansion of this condition.

9.11 Analytical Proofs of Geometric Theorems

We can prove many theorems of plane geometry with the help of the formulae coordinate geometry. We consider some examples.

Example 9.11

Prove that the line segment joining the mid-points of two sides of a triangle parallel to the third side and equal to one-half its length.

Solution

Consider for triangle OPQ with vertices at $O(0, 0)$, $P(a, 0)$ and $Q(b, c)$ (See Fig. 9.10)

The mid-point A of OQ is $\left(\frac{b}{2}, \frac{c}{2}\right)$. The mid-point B of PQ is $\left(\frac{a+b}{2}, \frac{c}{2}\right)$. The slop m_1 of AB is given by

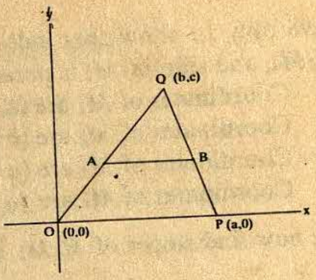


Fig. 9.10

$$m_1 = \frac{\frac{c}{2} - \frac{c}{2}}{\left(\frac{a+b}{2} - \frac{b}{2}\right)} = 0$$

The slope m_2 of OP is given by

$$m_2 = \frac{0-0}{a-0} = 0$$

Thus, $m_1 = m_2$.
Hence, the line joining the mid-points of the two sides is parallel to the third side.

Also, $$AB = \sqrt{\left(\frac{a+b}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2} = \frac{a}{2},$$

and $$OP = \sqrt{(0-a)^2 + (0-0)^2} = a$$

Hence, the line joining the two mid-points A and B is equal to one-half of the base.

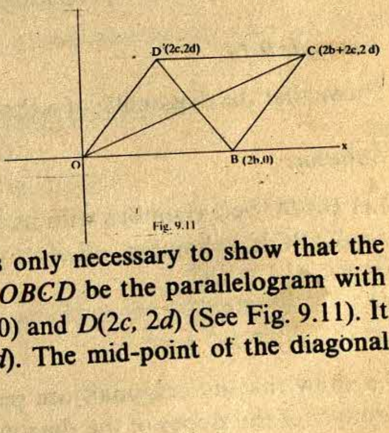


Fig. 9.11

Example 9.12

Prove that the diagonals of a parallelogram bisect each other.

Solution

To prove that the diagonals bisect each other, it is only necessary to show that the mid-point of each diagonal is the same point. Let $OBCD$ be the parallelogram with three of its vertices at the points $O(0, 0)$, $B(2b, 0)$ and $D(2c, 2d)$ (See Fig. 9.11). It may be seen that the fourth vertex is $C(2b+2c, 2d)$. The mid-point of the diagonal OC is $(b+c, d)$.

This completes the proof.

Example 9.13

Prove that the segments joining the mid-points of the adjacent sides of a quadrilateral form a parallelogram.

Solution

Let the vertices of the quadrilateral be at the points $O(0, 0)$, $A(2a, 0)$, $B(2b, 2c)$, $C(2d, 2e)$ (See Fig. 9.12). Let M_1, M_2, M_3 and M_4 represent the mid-points of OC, CB, BA and OA respectively. To prove that the quadrilateral $M_1M_2M_3M_4$ is a parallelogram, we have only to show that side M_2M_3 is parallel to side M_1M_4, and side M_1M_2 is parallel to M_4M_3.

Coordinates of M_1 are (d, e)
Coordinates of M_2 are $(b+d, c+e)$
Coordinates of M_3 are $(a+b, c)$
Coordinates of M_4 are $(a, 0)$

We now find slopes of M_1M_2, M_2M_3, M_4M_3 and M_4M_1

$$\text{Slope of } M_1M_2 = \frac{c+e-e}{b+d-d} = \frac{c}{b}$$

$$\text{Slope of } M_2M_3 = \frac{c-(c+e)}{a+b-(b+d)} = \frac{-e}{a-d}$$

$$\text{Slope of } M_4M_3 = \frac{0-c}{a-(a+b)} = \frac{c}{b}$$

$$\text{Slope of } M_4M_1 = \frac{0-e}{a-d} = \frac{-e}{a-d}$$

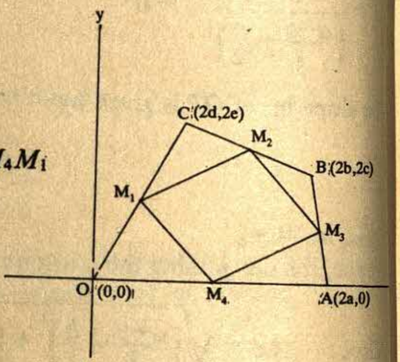


Fig. 9.12

We see that slope of M_1M_2 = slope of M_4M_3 and slope of M_2M_3 = slope of M_4M_1. Thus, $M_1M_2M_3M_4$ is a parallelogram.

Example 9.14

Prove that the diagonals of a rhombus are perpendicular to each other.

Solution

Let $OABC$ be a rhombus with its vertices as $(0, 0)$, $(x_1, 0)$, $(x_1 + x_2, y_2)$ and (x_2, y_2). As $OABC$ is a rhombus, all of its sides are equal.
Hence, $OA = OC$
or $\quad OA^2 = OC^2$
i.e. $\quad x_1^2 = x_2^2 + y_2^2 \qquad$ ...(i)

To show that its diagonals are perpendicular to each other, we shall show that the product of the slopes of the diagonals is -1.

Now slope of $OB = \dfrac{y_2}{x_1 + x_2}$

and slope of $AC = \dfrac{y_2}{x_2 - x_1}$

Hence, the product of the slopes $= \dfrac{y_2}{x_1 + x_2} \cdot \dfrac{y_2}{x_2 - x_1}$

$$= \frac{y_2^2}{x_2^2 - x_1^2}$$

$$= \frac{y_2^2}{-y_2^2} = -1 \qquad \{\text{ using } (i)\}$$

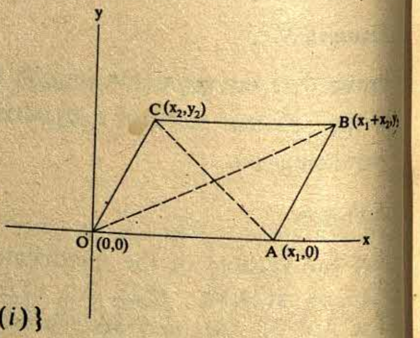


Fig. 9.13

Hence, the diagonals of a rhombus are perpendicular to each other.

9.12 Distantce of a Point from a Line

Case 1: Let the equation of the line AB be

$$x \cos \alpha + y \sin \alpha - p \qquad \text{...(i)}$$

Let P be the point (x', y') and d be the length of the perpendicular NP, P being assumed to lie on the opposite side of the line AB from O.

Draw $A'PB'$ parallel to AB through P, and draw the common perpendicular OTT' from O on AB and $A'B'$.

Then $OT = p$ and the angle $AOT = \alpha$,

The perpendicular from O on $A'B'$ is of length OT', and makes an angle α with OX; hence the equation of $A'B'$ is

$$x \cos \alpha + y \sin \alpha = OT' = p + d$$

The coordinates (x', y') of P satisfy the equation of $A'B'$

$$\therefore\ x' \cos \alpha + y' \sin \alpha = p + d$$

$$\text{or}\quad d = x' \cos \alpha + y' \sin \alpha - p$$

Hence, the length of the perpendicular is the result of substituting the coordinates of P in the expression $x \cos \alpha + y \sin \alpha - p$.

Case II: Let the equation of the line be given in the general form

$$Ax + By + C = 0 \qquad \text{... (ii)}$$

Reducing the general equation to the normal form, we have

$$\frac{-Ax}{\sqrt{A^2+B^2}} - \frac{By}{\sqrt{A^2+B^2}} = \frac{C}{\sqrt{A^2+B^2}}$$

Now, the length of the perpendicular segement drawn from the given point (x', y') to (ii) is

$$\frac{-Ax' - By' - C}{\sqrt{A^2+B^2}}$$

$$= -\frac{Ax' + By' + C}{\sqrt{A^2+B^2}}$$

Neglecting the negative sign as the length of a segement is always positive, we have the length of the perpendicular segment as

$$\left| \frac{Ax' + By' + C}{\sqrt{A^2+B^2}} \right| \qquad (9.13)$$

Example 9.15

Find the distance between the line $3x - 4y + 12 = 0$ and the point $(4, 1)$.

Solution

Since A = 3, B = −4, C = 12, $x_1 = 4$ and $y_1 = 1$, by the formula for the distance between a line and a point, we have

$$d = \left|\frac{3\times4+(-4)\times1+12}{\sqrt{(3)^2+(-4)^2}}\right|$$

$$= \left|\frac{12-4+12}{\sqrt{25}}\right| = \frac{20}{5} = 4$$

Example 9.16

Find the perpendicular distance of the point (a, b) from the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution

The required distance is

$$d = \frac{\frac{a}{a}+\frac{b}{b}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

$$= \frac{ab}{\sqrt{a^2+b^2}}.$$

9.13 Translation of Axes

Let (x, y) be the coordinates of any point P referred to the axes OX and OY. Let $O'X'$ and $O'Y'$ be the new axes parallel to OX and OY. Draw OH and HO', the abscissa and ordinate of O' referred to OX and OY. Let the coordinates of O' referred to the old axes be (h, k) so that $OH = h$, $HO' = k$. OM and MP are the abscissa and ordinate of the point P referred to the old axes i.e. $OM = x$ and $MP = y$. $O'M'$ and $M'P$ are the abscissa and ordinate of P referred to the new axes $O'X'$ and $O'Y'$. Let $O'M' = x'$ and $M'P = y'$. If we want to transform an equation in x and y into corresponding equation in x' and y', we will have to express the old coordinates x, y in terms of the new ones x', y' so that the transformation can be performed by direct substitution. It is easily seen from the figure that

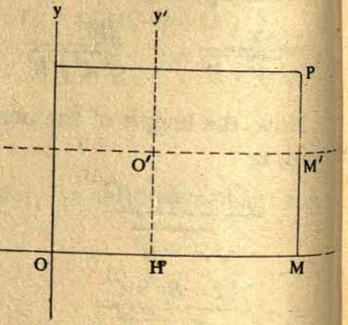


Fig. 9.15

$$OM = h + O'M';\ MP = k + M'P,$$

or

$$x = x' + h,\ y = y' + k \qquad \ldots(9.14)$$

We have, therefore, to write $(x' + h)$ for x and $(y' + k)$ for y in the equation which we wish to transform. We thus get an equation in x', y'. So if the locus of P

with respect to OX and OY be $f(x, y) = 0$, the equation to the same locus when the origin is transferred to O, the axes retaining their directions, becomes $f(x'+h, y'+k) = 0$ where x', y' are the current coordinates with reference to the new axes.

We can easily check that the area of a triangle (or a quadrilateral) and a slope of a line remain unaffected with the change of axes i.e. the area of a triangle and slope of a line are invariant under the translation of axes.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC referred to some rectangular coordinate axes. Let the origin be shifted to the point (h, k) with axes retaining their directions. Then the area Δ of the triangle ABC with respect to the old coordinate axes is given by.

$$\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right] \quad \text{... (i)}$$

Let (x_1', y_1'), (x_2', y_2') and (x_3', y_3'), be the coordinates of A, B, C with respect to the new axes. Then we know that

$$x_1 = x_1' + h,\ y_1 = y_1' + k$$
$$x_2 = x_2' + h,\ y_2 = y_2' + k$$
$$x_3 = x_3' + h,\ y_3 = y_3' + k$$

Now substituting these values in (i), we get

$$\Delta = \frac{1}{2}\Big[(x_1' + h)(y_2' + k - y_3' - k) + (x_2' + h)(y_3' + k - y_1' - k) + (x_3' + h)\times$$
$$(y_1' + k - y_2' - k) \Big]$$
$$= \frac{1}{2}\left[x_1'(y_2' - y_3') + x_2'(y_3' - y_1') + x_3'(y_1' - y_2') \right]$$

which is nothing but the area of the triangle with respect to the new coordinate axes. Thus the area with reference to both the coordinate axes remains the same.

Similarly, we can prove that the slope of a line does not change with the change of axes (parallel translation).

For if

$$Ax + By + C = 0 \quad \text{...(ii)}$$

is the equation of a straight line referred to the old coordinate axes, then the slope of the line is seen to be equal to

$$m = -\frac{A}{B}$$

Now, if the origin is shifted to the point (h, k), then as before any point (x', y') on the straight line with respect to the new coordinate axes will satisfy the following relations.

$$x = x' + h \text{ and } y = y' + k.$$

Therefore the equation of the straight line can be written as

$$A(x' + h) + B(y' + k) + C = 0$$

or $\quad Ax' + By' + (Ah + Bk + C) = 0 \quad \ldots$ (iii)
which is the equation of the straight line in the new coordinate axes. Now, we can easily see that the slope of the straight line given by (iii) is

$$m' = -\frac{A}{B}$$

which is same as m. Therefore, we have established that the slope of a straight line is invariant under the translation of axes.

In fact, as we have learned earlier, all the geometric properties in Euclidean Geometry remain unchanged under rigid motion. Since parallel translation of axes is only a particular case of rigid motion (which includes rotation also), it is quite obvious that all geometric properties shall remain unchanged when we transform the coordinates in this way.

We shall see later that translation of axes provides a very useful tool for obtaining equations of different loci in simple form, or for providing simple proofs of geometric properties.

EXERCISE 9

Find the equation of the line in each of the problems 1 to 4.

1. Through (4, 3) with slope 2.
2. Through (0, −2) with slope −4.
3. Through (0, −3) and (5, 0)
4. Through (−1, −2) and (−5, −2)
5. Find the lines through the point (0, 2) making an angle $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ with x-axis. Also the lines parallel to them cutting the y-axis at a distance 2 below the origin. Find also their point of intersection with x-axis.
6. What are the inclinations to the x-axis of the lines $y = \frac{1}{3}x\sqrt{3}+3$ and $y = \sqrt{3}x + 3$?
   Show that the line $y = x + 3$ bisects the angle between them.
7. Find the equation of the line that has y-intercept 4 and is parallel to the line $2x - 3y = 7$.
8. Find the equation of the line that has x-intercept -3 and is perpendicular to the line $3x + 5y = 4$.
9. Find the equation of a straight line passing through the point (2, 2), such that the sum of its intercepts on the axes = 9.
10. The vertices of a triangle are the points (0, 0), (2, 4) and (6, 4). Find the equations of its sides.
11. The mid-points of the sides of a triangle are (2, 1), (−5, 7), (−5, −5). Find the

equations of the sides.

12. Find the equation of the line that is parallel to $2x + 5y = 7$ and passes through the mid-point of the line joining (2, 7) and (−4, 1).
13. Find the equation of the line that is perpendicular to $3x + 2y = 8$ and passes through the mid-point of the line joining (5, −2) and (2, 2).
14. Find the equation of the straight line bisecting the segment joining the points (5, 3) and (4, 4) and making an angle of 45° with the x-axis.
15. If p be the measure of the perpendicular segment from the origin on the line whose intercepts on the axes are a and b, show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

16. Obtain the perpendicular form of the equation of lines from the given values of p and w.

(i) $p = 3, w = 45°$ (ii) $p = 5, w = 30°$
(iii) $p = 5, w = 135°$ (iv) $p = 1, w = 90°$

17. Reduce each of the following to the perpendicular form and find p.

(i) $x + y - 2 = 0$ (ii) $4x + 3y - 9 = 0$
(iii) $x - 4 = 0$ (iv) $y - 2 = 0$

18. Which of the lines $2x - y + 3 = 0$ and $x - 4y - 7 = 0$ is farther from the origin?
19. The perpendicular distance of a line from the origin is 5 cm and its slope is −1. Find the equation of the line.
20. Show that the origin is equidistant from the three straight lines, $4x + 3y + 10 = 0$, $5x - 12y + 26 = 0$, $7x + 24y = 50$
21. Find the angle between the straight line $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.
22. Find the equation of the lines through the origin making angle of 60° with the line $x + y\sqrt{3} + 3\sqrt{3} = 0$, also the coordinates of points where they meet the line.

Classify the pairs of lines in Exercise 23 to 25 as coincident, parallel, or intersecting.

23. $6x + 14y - 16 = 0$, $12x + 28y - 32 = 0$
24. $3x - 4y = 8$, $3x + 4y = 11$
25. $x - 2y = 7$, $4y - 2x = 13$

Find the distance between the line and the point in each of the following Exercises from 26 to 28.

26. $4x + 3y - 5 = 0$, (−2, −1)
27. $5x + 12y - 41 = 0$, (3, 0)
28. $y = 4$, (2, 3)

CHAPTER 10

FAMILY OF LINES

10.1 Equation of Family of Lines

Let $A_1x + B_1y + C_1 = 0$ ...(10.1)

and $A_2x + B_2y + C_2 = 0$ ...(10.2)

be two given lines, Then

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0 \qquad ...(10.3)$$

represents for all values of k, some straight lines passing through the intersection of $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$. Because the values of x and y which satisfy (10.1) and (10.2) simultaneously, obviously satisfy (10.3) also. Hence (10.3) represents a line through the point of intersection of (10.1) and (10.2) for every k. Hence, (10.3) represents the *family of lines* passing through the point of intersection of the two lines.

Example 10.1

Find the equation of the straight line parallel to the y-axis and drawn through the point of intersection of $x-7y+5 = 0$ and $3x+y-7 = 0$.

Solution

The equation of any straight line through the point of intersection is of the form

$$x-7y+5 + k(3x+y-7) = 0$$

or

$$(1+3k)x + (k-7)y + 5-7k = 0$$

If this line is parallel to y-axis, the coefficient of y is zero, hence $k = 7$, and the equation becomes

$$x-2=0.$$

Example 10.2

Find the equation of the line through the intersection of $3x+4y = 7$ and $x-y+2 = 0$, and with slope 5.

Solution

The equation of the family of lines will be

$$(3x+4y-7)+k(x-y+2)=0$$

or
$$(3+k)\,x+(4-k)y+(2k-7)=0$$

The slope of each member of the family is

$$\frac{k+3}{k-4}$$

But slope for the line is given to be 5. Hence,

$$\frac{k+3}{k-4}=5$$

or
$$k+3=5k-20,\text{ i.e., }k=\frac{23}{4}$$

Therefore, the equation of the desired line is

$$(3x+4y-7)+\frac{23}{4}(x-y+2)=2$$

or
$$4(3x+4y-7)+23(x-y+2)=0$$

or
$$35x-7y+18=0.$$

EXERCISE 10.1

1. Find the equations of the lines through the point of intersection of $x+2y=5$ and $x-3y=7$, and passing through the point
   (i) $(0, 0)$ (ii) $(2, -3)$
   (iii) $(1, 0)$ (iv) $(0, -1)$.
2. Find the equations of the lines through the point of intersection of $5x-3y=1$ and $2x+3y=23$, and perpendicular to the line whose equation is
   (i) $x-2y=3$ (ii) $x=0$
   (iii) $y=0$ (iv) $5x-3y=1$
3. Find the equation of the line through the intersection of the lines $x+2y-3=0$ and $4x-y+7=0$ and which is parallel to $5x+4y-20=0$.
4. Find the equation of the line through the intersection of the lines $2x+3y-4=0$ and $x-5x+7=0$ that has its x-intercept equal to -4.

10.2 Pair of Straight Lines Through Origin

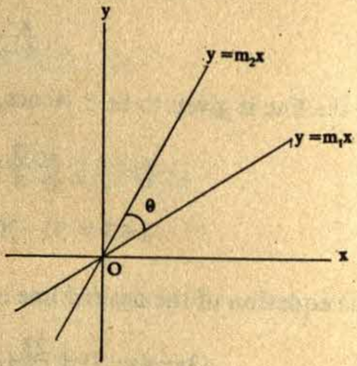
We shall show that the homogeneous equation of the second degree

$$ax^2+2hxy+by^2=0$$

represents a pair of straight lines passing through the origin if $h^2 \geq ab$.

Solving the above equation as a quadratic equation for x, we get

$$x = \frac{-h \pm \sqrt{h^2-ab}}{a}\, y, \quad (h^2 \geq ab)$$



i.e. either $ax + (h + \sqrt{h^2-ab})y = 0$

or $\quad ax + (h - \sqrt{h^2-ab})y = 0$

Fig.10.1

Each of the above is a straight line passing through the origin.

Hence, the homogeneous equation

$$ax^2+2hxy+by^2 = 0$$

represents two straight lines passing through the origin. These lines are real and distinct, if $h^2 > ab$, coincident if $h^2 = ab$, the lines do not exist if $h^2 < ab$.

10.3 Angle Between the Pair of Straight Lines

Let the pair of straight lines be reqresented by the equation $ax^2+ 2hxy+ by^2 = 0$.

Since the above equation represents a pair of straight lines passing through the origin, they will be given by the equations

$$y = m_1x \qquad ...(10.4)$$

and

$$y = m_2x \qquad ...(10.5)$$

or

$$y - m_1x = 0 \text{ and } y - m_2x = 0$$

Then

$$(y - m_1x)\,(y - m_2x) = 0$$

or

$$m_1m_2x^2-(m_1+m_2)xy+y^2 = 0$$

This equation is the same as

$$ax^2+2hxy+by^2 = 0.$$

Therefore, comparing the coefficients,

$$\frac{m_1m_2}{a} = -\frac{(m_1+m_2)}{2h} = \frac{1}{b}$$

This gives

$$m_1m_2 = \frac{a}{b} \qquad ...(10.6)$$

and $m_1 + m_2 = -\frac{2h}{b}$ (10.7)

If θ be the angle between the given straight lines, then

$$\tan\theta = \frac{m_1 - m_1}{1 + m_1 m_2}$$
$$= \frac{\sqrt{(m_1+m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

Substituting the values of $m_1 m_2$ and $m_1 + m_2$, we obtain

$$\tan\theta = \frac{\sqrt{4h^2 - 4ab}}{a+b} \quad \text{...(10.8)}$$

Hence,

$$\theta = \tan^{-1}\left\{\frac{2\sqrt{h^2 - ab}}{a+b}\right\} \quad \text{...(10.9)}$$

Notice that the condition for coincidence of the lines is

$$h^2 = ab \quad \text{...(10.10)}$$

In other words, for the coincidence of lines, the expression $ax^2 + 2hxy + by^2$ should be a perfect square.

The condition for perpendicularity is

$$a + b = 0 \quad \text{...(10.11)}$$

10.4 Equations of the Bisectors of the Angles

Let the lines be represented by the equation $ax^2 + 2hxy + by^2 = 0$

Suppose the lines are

$y - m_1 x = 0$, $y - m_2 x = 0$

Since from any point on a bisector the perpendicular distances on the lines are equal, we have

$$\frac{y - m_1 x}{\sqrt{1 + m_1^2}} = \pm \frac{y - m_2 x}{\sqrt{1 + m_2^2}}$$

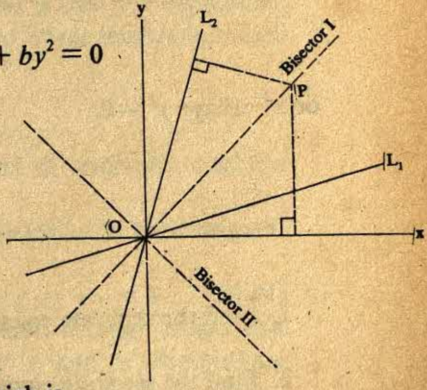


Fig.10.2

The two bisectors can be expressed in one equation which is

$$\left(\frac{y - m_1 x}{\sqrt{1 + m_1^2}} + \frac{y - m_2 x}{\sqrt{1 + m_2^2}}\right)\left(\frac{y - m_1 x}{\sqrt{1 + m_1^2}} - \frac{y - m_2 x}{\sqrt{1 + m_2^2}}\right) = 0$$

or $(1 + m_2^2)(y - m_1 x)^2 - (1 + m_1^2)(y - m_2 x)^2 = 0$

or $x^2(m_1^2 - m_2^2) - 2xy(m_1 - m_2)(1 - m_1 m_2) + y^2(m_2^2 - m_1^2) = 0$

or $x^2 - y^2 = 2xy\,\frac{1 - m_1 m_2}{m_1 + m_2}$ (since $m_1 - m_2 \neq 0$)

Substituting $\frac{a}{b}$ for m_1m_2 and $-\frac{2h}{b}$ for $m_1 + m_2$ as in section 10.3, equations (10.6) and (10.7), the above equation becomes

$$x^2-y^2 = 2xy\,\frac{a-b}{2h}$$

i.e. $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

which is the required equation of the bisectors for $ax^2+2hxy+by^2 = 0$.

Example 10.3

Find the equation of the lines bisecting the angles between the pair of lines

$$3x^2+xy-2y^2 = 0.$$

Solution

Using the formula $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$,

we get the equation

$$\frac{x^2-y^2}{3-(-2)} = \frac{xy}{\frac{1}{2}}$$

or $x^2-10xy-y^2 = 0$.

EXERCISE 10.2

1. Find the separate equations of the straight lines whose joint equation is $$x^2-5xy+6y^2 = 0$$
2. Find the separate equation of the straight lines whose joint equation is $$ab(x^2-y^2) + (a^2-b^2)xy = 0$$
3. Find the angle between the lines whose joint equation is $2x^2-3xy-6x^2 = 0$.
4. Find the straight lines represented by the equation $$y^2-xy-6x^2 = 0$$ and find the angle between them.
5. Prove that the angle between the straight lines given by $$(x\cos\alpha - y\sin\alpha)^2 = (x^2+y^2)\sin^2\alpha$$ is 2α.
6. Show that the bisectors of the angles between the lines $$(ax+by)^2 = 3(bx-ay)^2$$

are respectively parallel and perpendicular to the line $ax+by+c=0$.

10.5 Condition for the General Equation of Second Degree to Represent Two Straight Lines

We have seen that the general equation of the first degree in x and y, viz. $Ax+By+C=0$ represents a straight line. Let us now consider the product equation.

$$(Ax+By+C)(A'x+B'y+C')=0 \qquad \ldots(10.12)$$

and examine what locus is represented by this.

Since the two factors on the left hand side of equation (10.12) are $Ax+By+C$ and $A'x+B'y+C'$, the equation is obviously satisfied if either of these factors is equal to zero. Hence, equation (10.12) represents a pair of straight lines

$$Ax+By+C=0 \text{ and } A'x+B'y+C'=0$$

If we multiply the two factors on the left hand side of (10.12), we get an equation of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \ldots(10.13)$$

where the values of a, b, c, f, g, h are easily determined in terms of A, B, C, A', B', C'. This is the most general equation of the second degree and will represent a pair of straight lines provided the coefficients of various powers of x and y and the constant term are suitably determined. The obvious condition is that the expression on the left hand side of equation (10.13) should break up into linear factors in x and y.

Let the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represent a pair of straight lines and let (x_1, y_1) be the point of intersection of the lines.

If the origin is transferred to (x_1, y_1) the axes remaining parallel to their original directions, equation (10.13) transforms into

$$a(x+x_1)^2+2h(x+x_1)(y+y_1)+b(y+y_1)^2+2g(x+x_1)+2f(y+y_1)+c=0 \;\ldots.\; (10.14)$$

Referred to new axes,(10.14) is the equation of two straight lines through origin, and must therefore be a homogeneous quadratic equation in x and y. Simplifying further, we have $ax^2+2hxy+by^2+2(ax_1+hy_1+g)x+2(hx_1+by_1+f)y+ax_1^2$

$$+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \qquad \ldots(10.15)$$

Hence,(10.15) is a homogeneous quadratic equation in x and y with respect to new coordinate axes. The pair of straight lines passes through origin. Hence, the coefficients of x and y and the constant in (10.15) must separately vanish.

Thus
$$ax_1+hy_1+g=0 \qquad \ldots(10.16)$$
$$hx_1+by_1+f=0 \qquad \ldots(10.17)$$
and
$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \qquad \ldots(10.18)$$

Multiplying (10.16) by x_1, (10.17) by y_1, adding and subtracting the result from (10.18), we get

$$gx_1+fy_1+c=0 \qquad \ldots(10.19)$$

Eliminating x_1, y_1 from (10.16), (10.17), and (10.19), we obtain,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

which is the condition that the equation(10.13)) should represent a pair of straigh lines.

Expanding the above determinant, the condition that the general equation of th second degree should represent two straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Note: The point of intersection can be found by solving any two of the equation (10.16), (10.17), and (10.19).

We can arrive at the same result by the following alternative method.

The equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines if the expression on the left can be broken up int two linear factors of the type

$$lx+my+n \text{ and } l'x+m'y+n'$$

We then have

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
$$\equiv (lx+my+n)\,(l'x+m'y+n')$$

which gives $ll' = a$, $mm' = b$, $nn' = c$

$$lm' + ml' = 2h$$
$$ln' + nl' = 2g, \ \ mn' + nm' = 2f$$

Multiplying the last three results together, we obtain

$$2ll'mm'nn' + ll'\,(m^2n'^2+n^2m'^2) + mm'(n^2l'^2 + l^2n'^2) + nn'(l^2m'^2+m^2l'^2) = 8\,fgh$$

which reduces to

$$2abc + a(4f^2-2bc) + b(4g^2-2ac) + c(4h^2-2ab) = 8fgh$$

or

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Remark

We have noted in the first proof above that if the equation $ax^2 + 2hxy + by^2 +$ $2gx + 2fy + c = 0$ represents a pair of lines, and if we translate the axes so that th point of intersection is taken as new origin, then this equation reduces to th homogeneous form

$$ax^2 + 2hxy + by^2 = 0$$

in new coordinates. We shall be using this fact quite often in what follows.

Sufficiency of the condition

We have seen that if the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{...(10.20)}$$

represents two straight lines, then

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \qquad \text{... (10.21)}$$

This shows that (10.21) is the necessary condition that the general equation should represent two straight lines. We shall now show that this condition is sufficient also.

If

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \text{ i.e. } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$$

we can find x_1 and y_1 to satisfy three linearly independent equations

$$ax_1 + hy_1 + g = 0$$
$$hx_1 + by_1 + f = 0$$
$$gx_1 + fy_1 + c = 0$$

If the origin is now transfered to (x_1, y_1) as determined by any two of the above equations, the general equation of the second degree is seen to reduce to

$$ax'^2 + 2hx'y' + by'^2 = 0$$

which is the equation to a pair of straight lines.

10.6 Angle between two Lines

If the origin in transferred to the point of intersection of the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ the equation reduces to

$$ax^2 + 2hxy + by^2 = 0$$

The lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are, therefore, parallel to the pair of lines

$$ax^2 + 2hxy + by^2 = 0$$

through the origin.

The angle between the given lines is, therefore,

$$\tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a + b}\right)$$

The lines are parallel if $h^2 = ab$, and perpendicular if $a + b = 0$.

EXERCISE 10.3

1. Find what the following equations become when the origin is transformed to the point (1, 1)
   (i) $x^2 + xy - 3x - y + 2 = 0$
   (ii) $xy - y^2 - x + y = 0$
   (iii) $xy - x - y + 1 = 0$
   (iv) $x^2 - y^2 - 2x + 2y = 0$

2. Show that the equations
$3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$
represents a pair of straight lines.
3. Show that the equation
$2x^2 - 5xy + 2y^2 - 3x + 3y + 1 = 0$
represents two straight lines intersecting at an angle $\tan^{-1}\frac{3}{4}$.
4. Show that the equation $x^2 - y^2 - x + 3y - 2 = 0$ represents a pair of straigh lines, find them, and show they are at right angles.
5. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair o straight lines, show that they intersect in the point

$$\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$$

6. Show that the four lines given by the equation
$$3x^2 + 8xy - 3y^2 = 0$$
and $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$
form a square. Find the equations of the diagonals of the square.

CHAPTER 11

CIRCLES AND FAMILY OF CIRCLES

11.1 Standard Form of the Equation of a Circle

As we have seen in section 8.7 that the locus is a set of points satisfying some geometrical conditions, and this set of points gives rise to a curve. Recall that the circle is a particular curve satisfying the condition that the distance of each of its points from a fixed point, always remains constant. Therefore, the precise definition of a circle is as follows:

Definition 1.1

A circle is the locus of a point which moves in such a way that its distance from a fixed point is always a positive constant.

The fixed point is ealled the centre and the given distance is called the radius of the circle.

Now, we obtain the equation of a circle, as under, with a given centre and radius.

Theorem 11.1

If the centre of the circle is at $C(h, k)$ and has radius r then the equation of the circle is given by

$$(x-h)^2+(y-k)^2=r^2 \qquad \text{...(11.1)}$$

Proof: Let $P(x, y)$ by any point on the circle (See Fig 11.1)

Join the point $P(x, y)$ with the centre $C(h, k)$. Then

$$|CP| = r.$$

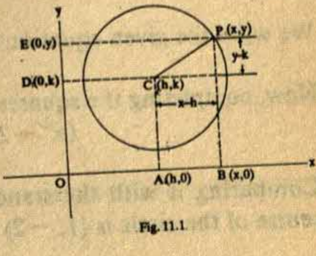


Fig. 11.1

Using the distance formula, we get

$|CP| = \sqrt{(x-h)^2+(y-k)^2}$

Therefore, $\sqrt{(x-h)^2+(y-k)^2}=r$

or $(x-h)^2+(y-k)^2=r^2.$

If $Q(x_1, y_1)$ is another point lying on this locus, then

$$(x_1 - h)^2 + (y_1 - k)^2 = r^2,$$

and therefore the distance $CQ = r$. The equation $(x - h)^2 + (y - k)^2 = r^2$ represents a circle having centre at the point $C(h, k)$ and radius equal to r.

Corollary: If the centre of the circle with radius r is at the origin O (0, 0), the equation of the circle reduces to the following form

$$x^2 + y^2 = r^2 \qquad ...(11.2)$$

The equation (11.1) is called the *standard form* of the equation of the circle. The circle is real if $r^2 > 0$. If $r^2 < 0$, then no point in the plane satisfies equation (11.1) and the graph of the equation is empty. For instance, we have no point in the plane which satisfies the equation $x^2 + y^2 = -1$.

If $r^2 = 0$, equation (11.1) becomes $(x - h)^2 + (y - k)^2 = 0$ and this gives $x = h$ and $y = k$. Why? Therefore, only one point satisfies the equation of such a circle and this point coincides with the centre. Such a circle is called a Point Circle. Thus a point circle is the circle whose radius is zero.

Example 11.1

Find the equation of the circle with centre (− 3, 2) and radius 4.

Solution

Here the point (h, k) is (− 3, 2) and radius 4. Hence, substituting $h = -3$, $k = 2$ and $r = 4$ in (11.1), we have

$$\left[x - (-3)\right]^2 + (y - 2)^2 = 4^2$$

or $\quad (x + 3)^2 + (y - 2)^2 = 16.$

Example 11.2

Find the centre and radius of the circle

$$x^2 + y^2 - 2x + 4y = 8$$

Solution

We write the given equation in the form

$$(x^2 - 2x) + (y^2 + 4y) = 8$$

Now, completing the squares within parentheses, we get

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 1 + 4$$
$$(x - 1)^2 + (y + 2)^2 = 13$$

Comparing it with the standard form of the equation of the circle, we see that the centre of the circle is (1, − 2) and radius is $\sqrt{13}$.

EXERCISE 11.1

1. Find the centre and radius of each of the following circles:
   (i) $x^2 + (y - 1)^2 = 2$
   (ii) $(x + 5)^2 + (y - 3)^2 = 30$
   (iii) $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{4}$
   (iv) $x^2 + y^2 - 4x + 6y = 5$
   (v) $x^2 + y^2 - x + 2y - 3 = 0$
2. Find the equation of each of the following circles:
   (i) Centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$
   (ii) Centre $(-3, -2)$ and radius 7
   (iii) Center $(0, -1)$ and radius 1
   (iv) Centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$
   (v) Centre (a, a) and radius $\sqrt{2}a$
   (vi) Centre $(a \cos \alpha, a \sin\alpha)$ and radius a
3. Find the equation of a circle of radius 5 whose centre lies on x- axis and passes through the point (2, 3).

11.2 General Form of the Equation of a Circle

Consider the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(11.3)$$

which contains terms of second degree except xy term, namely, x^2 and y^2, terms of first degree, namely, $2gx$ and $2fy$, and the term of zero degree, namely, the constant term c. Equation (11.3) is known as the general form of the equation of a circle.

We shall now show that the equation (11.3) represents a circle.

The given equation is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Adding $g^2 + f^2 - c$ on both sides of above equation, we get

$$x^2 + g^2 + 2gx + y^2 + f^2 + 2fy = g^2 + f^2 - c$$

or $$(x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

Comparing this equation with the standard form of the equation of a circle we get

$$h=-g,\ k=-f,\ a=\sqrt{g^2+f^2-c}$$

Hence, we see that equation (11.3) represents a circle with centre $(-g, -f)$ and radius $=\sqrt{g^2+f^2-c}\quad (g^2+f^2\geq c)$

Remark

The equation of a circle has the following properties:

(i) It is a second degree equation in x and y.

(ii) It contains no term of the form xy.

(iii) Coefficients of x^2 and y^2 are always equal.

To determine the coordinates of the centre of a circle (if the equation of a circle is given) we make the coefficients of x^2 and y^2 equal to 1, and then take the negative halves of the coefficients of x and y. For example, the centre of the circle $x^2+y^2-8x-12y-48=0$ is $(4, 6)$ and that of the circle $3x^2+3y^2+12x+18y-12=0$ is $(-2, -3)$.

Since equation (11.3) involves three constants, therefore at least three conditions are required to determine a circle and any given condition gives a relation among all the three constants. However, we may note that having the centre specified is equivalent to two conditions, since its coordinates determine two of the constants in these equations. The three conditions may take various forms. We illustrate some ot these in the following examples.

Example 11.3

Find the equation of the circle that passes through the points (1,0), (– 1,0), and (0,1).

Solution

Substituting the coordinates of the three points successively in equation (11.3), we get

$$\begin{aligned}2g+c&=-1\\-2f+c&=-1\\2f+c&=-1\end{aligned}$$

The solution of the these simultaneous equations results in $f=0, g=0$ and $c=-1$. Therefore, the equation of the circle passing through the three given points is $x^2+y^2=1$.

Example 11.4

Find the equation of the circle which passes through the points (20, 3), (19, 8) and (2,–9). Find its centre and radius.

Solution

By substitution of coordinates in the general equation of a circle, we have

$$\begin{aligned} 40g + 6f + c &= -409 \\ 38g + 16f + c &= -425 \\ 4g - 18f + c &= -85 \end{aligned}$$

From these three equations, we get

$g = -7, f = -3$ and $c = -111$.

Hence, the equation of the circle is

$$x^2 + y^2 - 14x - 6y - 111 = 0,$$

or $\quad (x - 7)^2 + (y - 3)^2 = 13^2$

Its centre is (7, 3) and radius is 13.

Example 11.5

Find the equation of the circle concentric with the circle $x^2 + y^2 - 4x - 6y - 9 = 0$ and passing through the point $(-4, -5)$.

Solution

Two circles having same centre are called concentric. Therefore, the centre of the required circle is the centre of the circle $x^2 + y^2 - 4x - 6y - 9 = 0$.

Hence, centre = (2, 3)

Now, the circle passes through the points $(-4, -5)$

$\therefore$ Radius of the circle $= \sqrt{(2 + 4)^2 + (3 + 5)^2} = 10$

$\therefore$ Equation of the circle is

$$(x - 2)^2 + (y - 3)^2 = 100$$

or $\quad x^2 - 4x + 4 + y^2 - 6y + 9 = 100$

or $\quad x^2 + y^2 - 4x - 6y - 87 = 0.$

EXERCISE 11.2

1. Find the equation of each of the following circles:
   - (i) Centre lies on the line $x - 4y = 1$, and passes through the points (3, 7) and (5, 5).
   - (ii) Passing through the points (2, 4) and centre at the intersection of the lines $x - y = 4$ and $2x + 3y = -7$.
   - (iii) Passing through the points $(2, -6)$, $(6, 4)$ and $(-3, 1)$.
   - (iv) Passing through the vertices of the triangle whose sides are along $x + y = 2$, $3x - 4y = 6$, and $x - y = 0$.
2. Find the equation of the circle
   - (a) Passing through (0, 0) and intercepting lengths a and b on the axes.

(b) Whose centre is (h, k) and which passes through the point (p, q).

3. Show that the points (5, 5), (6, 4), (− 2, 4) and (7, 1) all lie on a circle, and fi its equation, centre and radius.
4. Find the equation of the circle which passes through the centre of the circ $x^2 + y^2 + 8x + 10y - 7 = 0$ and is concentric with the circle $2x^2 + 2y^2 - 8x - 12y - 9 = 0$.

11.3 Equation of a Circle in Parametric Form

Let C be a circle centred at origin and let $P(x, y)$ be any point on it. Let the radiu OP of C be r, which makes an angle α with the positive direction of x-axis. Dra perpendicular PR from P on x-axis. Then we have

$$x = OR = r\cos\alpha$$

and $$y = RP = r\sin\alpha$$

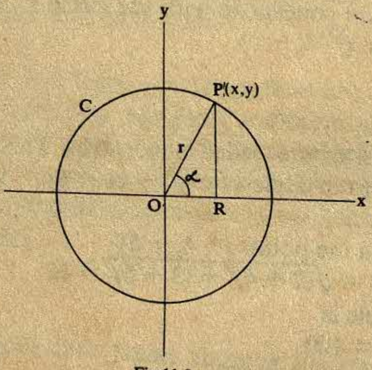


Fig.11.2

the coordinates of any point on the circle in terms of the variable α. The equation of the circle $x^2 + y^2 = r^2$ is satisfied for for all values of α which lies between 0 and 2π for $x = r\cos\alpha$ and $y = r\sin\alpha$

Thus $x = r\cos\alpha$, $y = r\sin\alpha$, $0 \le \alpha \le 2\pi$, is said to be parametric representation of the circle $x^2 + y^2 = r^2$ in terms of parameter α.

Example 11.6

Show that the point

$$x = \frac{2rt}{1+t^2}, y = \frac{r(1-t^2)}{1+t^2} \quad (r \text{ constant})$$

lies on a circle for all values of t such that $-1 \le t \le 1$. (This is also a parametric equation of a circle.)

Hint: Squaring and adding, we get $x^2 + y^2 = r^2$.

Example 11.7

Give the parametric form of the circle

$$x^2+y^2=25.$$

Solution

The equation of the circle is given to be

$$x^2+y^2=25$$

Here $r=5$. Hence the parametric equation in terms of θ is

$$x=5\cos\theta,\ y=5\sin\theta.$$

Example 11.8

Find the Cartesian equation of the curve

$$x=5+3\cos\alpha$$
$$y=7+3\sin\alpha$$

Solution

We write the given equation in the following form:

$$\cos\alpha=\frac{x-5}{3} \text{ and } \sin\alpha=\frac{y-7}{3}$$

Since $\sin^2\alpha+\cos^2\alpha=1$. Therefore,

$$\left(\frac{x-5}{3}\right)^2+\left(\frac{y-7}{3}\right)^2=1$$

or $\quad (x-5)^2+(y-7)^2=9$

which is the required equation and shows that the given equations represent a circle with centre (5, 7) and radius 3.

EXERCISE 11.3

1. Find the parametric representation of following circles:
    - (i) $3x^2+3y^2=4$
    - (ii) $x^2+y^2+2x-4y-4=0$
    - (iii) $x^2+y^2+px+py=0$
2. Find the equations of the following curves in Cartesian form. Wherever the curve is a circle, find its centre and radius.
    - (i) $x=3\cos\alpha,\ y=3\sin\alpha$

(ii) $x = a + c\cos\alpha,\ y = b + c\sin\alpha$
(iii) $x = 7 + 4\cos\alpha,\ y = -3 + 4\sin\alpha$
(iv) $x = \frac{t}{2} + 1,\ y = 2t - 1.$

11.4 Equation of a Circle when the End Points of a Diameter are Given

Several methods can be given to find the equation of the circle when the coordina of the end points of the circle are given. Here we describe one of them.

Let $P(x, y)$ be any point on the circle of radius r. Let RS be the diameter havi coordinates (x_1, y_1) and (x_2, y_2) and let $C(h, k)$ denote the centre of the circle. Slopes of the straight lines PR and PS are respectively

$$\frac{y-y_1}{x-x_1} \text{ and } \frac{y-y_2}{x-x_2}$$

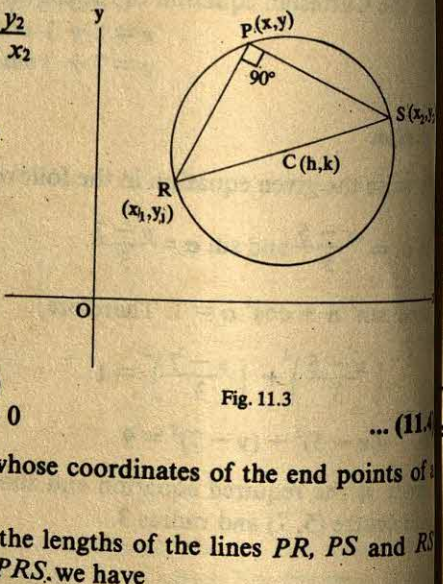


Fig. 11.3

Now, since the angle subtended at the point P in the semi-circle RPS is a right angle, therefore the two lines PR and PS are perpendicular and hence the product of their slopes must be -1 i.e.

$$\frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

or $\quad (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$

or. $\quad (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0 \qquad \ldots (11.$

which is the required equation of the circle whose coordinates of the end points of diameter are (x_1, y_1) and (x_2, y_2).

We can also find the equation by finding the lengths of the lines PR, PS and R and applying Pythagoras Theorem i.e. in $\Delta\ PRS$, we have

$$PR^2 + PS^2 = RS^2$$

Further, we can determine the equation of the circle if we find the coordinates o the centre and radius and then we apply the standard from.

Example 11.9

Find the equation of the circle, when the coordinates of the end points of whos diameter are (3,4) and (−3, −4).

Solution

Given $x_1 = 3, y_1 = 4; x_2 = -3, y_2 = -4$. Using equation (11.4), we get

$$(x-3)(x+3)+(y-4)(y+4)=0$$

or $\quad x^2-9+y^2-16=0$

or $\quad x^2+y^2=25$

which is the required equation of the circle.

EXERCISE 11.4

1. Find the equation of the circle when the coordinates of end points of the diameter are (p, q) and (r, s).
2. Find the equation of the circle drawn on the diagonal of the rectangle as its diameter whose sides are $x=6$, $x=-3$; and $y=3$ and $y=-1$.

11.5 Points of Intersection of a Line and a Circle with Centre at Origin, Condition of Tangency

Let the equation of the circle be

$$x^2+y^2=r^2 \qquad \text{...(11.5)}$$

and the equation of the line be

$$y=mx+c \qquad \text{...(11.6)}$$

To find the points of intersection of the circle (11.5) and the line (11.6), we have to solve them simultaneously.

Substituting the value of y from equation (11.6) in (11.5), we get

$$x^2+(mx+c)^2=r^2$$

or $\quad x^2+m^2x^2+c^2+2mxc=r^2$

or $\quad (1+m^2)x^2+2mcx+c^2-r^2=0 \qquad \text{...(11.7)}$

We see that the above equation is quadratic in variable x and, therefore, it will give two values of x or one value (of course repeating) of x or it will give no value of x. Then we get the corresponding values of y by substituting the values of x obtained from equation (11.7) in equation (11.6).

Thus every line intersects a circle in two points, in one point (in this case the line only touches the circle) and in no point.

Remark

In the last case the line lies somewhere outside the circle, will not touch the circle too.

Now, the equation (11.7) has two distinct roots, one repeated root or no real root according as the discriminant of the equation (11.7) satisfies

$$4m^2c^2-4(1+m^2)(c^2-r^2)>0$$

or $4m^2c^2-4(1+m^2)(c^2-r^2)=0$

or $4m^2c^2-4(1+m^2)(c^2-r^2)<0$

or $4r^2(1+m^2)>4c^2$

$4r^2(1+m^2)=4c^2$

$4r^2(1+m^2)<4c^2$,

or $r^2>\frac{c^2}{1+m^2} \Rightarrow r>\left|\frac{c}{\sqrt{1+m^2}}\right|$ ...(11.

$r^2=\frac{c^2}{1+m^2} \Rightarrow r=\left|\frac{c}{\sqrt{1+m^2}}\right|$ ...(11.

$r^2<\frac{c^2}{1+m^2} \Rightarrow r<\left|\frac{c}{\sqrt{1+m^2}}\right|$ ...(11.1

As the radius is always positive, we take the positive value of r.

Hence, a line intersects a circle in two distinct points, in one point or in no re point according to (11.8) (11.9) and (11.10) respectively.

Condition of Tangency

Definition 11.2

A line which meets a circle only in one point is called a tangent to the circle or other words we say that a line is a tangent to a circle if it only touches the circle.

Therefore, from equation (11.9), we get the following condition of tangency.

$$r=\frac{c}{\sqrt{1+m^2}} \quad ...(11.1$$

or, equivalently,

$$c=\pm r\sqrt{1+m^2} \quad ...(11.1$$

Remark:

The lines $y=mx\pm r\sqrt{1+m^2}$ are always tangent to the circle $x^2+y^2=r^2$.

Example 11.10

Find the coordinates of the points of intersection of the line $-x+y=1$ and t circle $x^2+y^2=25$.

Solution

The equation of the circle is

$$x^2+y^2=25$$

and the equation of the line is

$-x+y = 1$ or $y = 1+x$

Substituting this value of y in the equation of the circle,we get

$$x^2+(x+1)^2 = 25$$

or $\quad x^2+x^2+1+2x = 25$

or $\quad 2x^2+2x-24 = 0$

or $\quad x^2+x-12 = 0$

or $\quad x^2-3x+4x-12 = 0$

or $\quad x(x-3)+4(x-3) = 0$

or $\quad (x-3)(x+4) = 0$

from which if follows that $x = 3$ or $x = -4$. Putting these values in the equation $-x+y = 1$, we get the corresponding values of y as $y = 4$ or -3. Therefore, the points of intersection are $(3, 4)$ and $(-4, -3)$.

Example 11.11

Find the condition for a line $3x+4y-p = 0$ to be the tangent to the circle $x^2+y^2-64 = 0$.

Solution

We know that the condition of tangency is

$$c = \pm r\sqrt{1+m^2}$$

The given line is

$$3x+4y-p = 0 \quad \text{or} \quad y = -\frac{3}{4}x+\frac{p}{4}$$

$$\therefore c = \frac{p}{4}.$$

The radius of the circle is 8

$$\therefore \frac{p}{4} = \pm 8\sqrt{1+\left(\frac{-3}{4}\right)^2} = \pm\ 8\sqrt{\frac{25}{10}} = \pm\left(8\times\frac{5}{4}\right) = \pm 10$$

i.e. $p-40 = 0$ and $p+40 = 0$
which are the required conditions.

EXERCISE 11.5

1. Find the coordinates of the point of intersection of the straight line $7x-y-25 = 0$ and the circle $x^2+y^2=25$.
2. Find the point of intersection of the line $y = mx+c$ and the circle $x^2+y^2 = a^2$.

Also find the condition(s) when the line is tangent to the circle.

3. Find the coordinates of the points of intersection of the line $x+y = 2$ with circle $x^2+y^2 = 4$.
4. For what value of c will the line $y = 2x+c$ be a tangent to the circle $x^2+y^2 = 5$.

11.6 Equation of a Tangent to a Circle and Length of the Tangent

Before we find the equation of a tangent to circle, we need to define the tangent t curve.

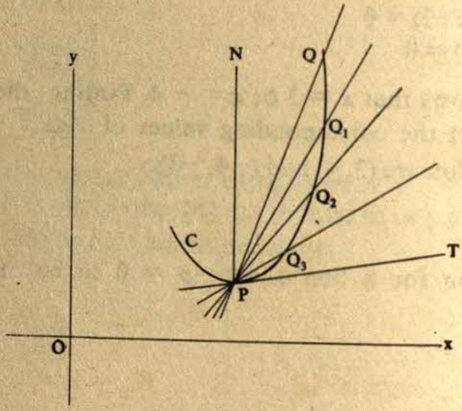


Fig.11.4

Let C be a curve. Let P and Q be any two points on the curve C as shown Fig. 11.4. Draw the second line PQ. Now, as the point Q moves along the cur towards P, the second line PQ turns about P (rotates about P), and Q approach P, the line PQ coincides with the limiting line PT. This limiting line PT is called th *tangent line* to the curve C at the point P.

The line perpendicular to the tangent to the curve C is called the *normal* to th curve C.

Now we find the equation of the tangent as under:

Theorem 11.2

The equation of the tangent to the circle $x^2+y^2 = r^2$ at a point (x_1, y_1) is given by

$$xx_1+yy_1 = r^2 \quad ...(11.1$$

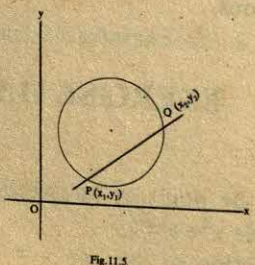


Fig.11.5

Proof: Let (x_1, y_1) be the coordinates of a given point on the circle $x^2+y^2=r^2$. Then this point must satisfy the equation i.e.

$$x_1^2+y_1^2=r^2 \qquad \text{...(11.14)}$$

Take another point $Q(x_2, y_2)$ on the circle. Then

$$x_2^2+y_2^2=r^2 \qquad \text{...(11.15)}$$

Now, the equation of PQ by the two–point formula will be

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \text{...(11.16)}$$

Subtracting (11.15) from (11.14), we have

$$y_1^2-y_2^2=x_2^2-x_1^2$$

or

$$(y_1-y_2)(y_1+y_2)=(x_2-x_1)(x_2+x_1)$$

Therefore,

$$\frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1}{y_2+y_1} \qquad \text{...(11.17)}$$

Since the tangent to the curve is the limiting position of the chord PQ as Q tends to P, therefore, as Q tends to P, i.e. as $x_2 \to x_1$ and $y_2 \to y_1$, we have

$$\frac{y_2-y_1}{x_2-x_1}=-\frac{2x_1}{2y_1}=-\frac{x_1}{y_1}$$

and equation (11.16) reduces to

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

or

$$xx_1-x_1^2+yy_1-y_1^2=0$$

or

$$xx_1+yy_1=x_1^2+y_1^2 \qquad \text{...(11.18)}$$

From (11.14), $x_1^2+y_1^2=r^2$. Hence, the above above equation gives

$$xx_1+yy_1=r^2$$

which is the required equation of the tangent to the circle $x^2+y^2=r^2$ at the given point (x_1, y_1).

If the tangent is desired at any point, say (x_1, y_1), of the circle given by the general equation,

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{...(11.19)}$$

we can proceed similarly as above. The equation of the tangent will be

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0. \qquad \text{...(11.20)}$$

[The proof is left for the reader.]

Remark

It may be noted that the equation of the tangent at (x_1, y_1) is, therefore, obtain from that of the equation of the circle itself by substituting xx_1 for x^2, yy_1 for $\frac{x+x_1}{2}$ for x and $\frac{y+y_1}{2}$ for y respectively.

Example 11.12

Find the equation of the tangent to the circle $x^2+y^2 = 13$ at the point (2, 3).

Solution

The equation of the tangent to the circle

$$x^2+y^2 = a^2 \text{ at } (x_1, y_1) \text{ is}$$

$$xx_1+yy_1 = a^2$$

Substituting $x_1=2$, $y_1 = 3$ and $a^2 = 13$, the equation of the tangent to the give circle is

$$2x+3y = 13.$$

Example 11.13

Find the equation of the tangent to the circle

$$x^2+y^2-26x+12y+105 = 0$$

at the point (7, 2).

Solution

Comparing the above equation of the circle with the standard equation of the circl we have

$$2g = -26,\ 2f = 12,\ c = 105$$

Also,

$$x_1 = 7,\ y_1 = 2$$

Therefore, from (11.20), the equation of the tangent is

$$7x + 2y - \frac{26}{2}(x+7) + \frac{12}{2}(y+2)+105 = 0$$

or

$$7x+2y-13(x+7) + 6(y+2) + 105 = 0$$

or

$$7x+2y-13x-91+6y+12+105 = 0$$

or

$$-6x+8y+26 = 0$$

or

$$3x-4y-13 = 0$$

Length of the Tangent

Let (x_1, y_1) be the given point P, and let the equation of the given circle $x^2+y^2= r^2$. Then by definition of the tangent line

$$\angle OTP = \frac{\pi}{2}$$

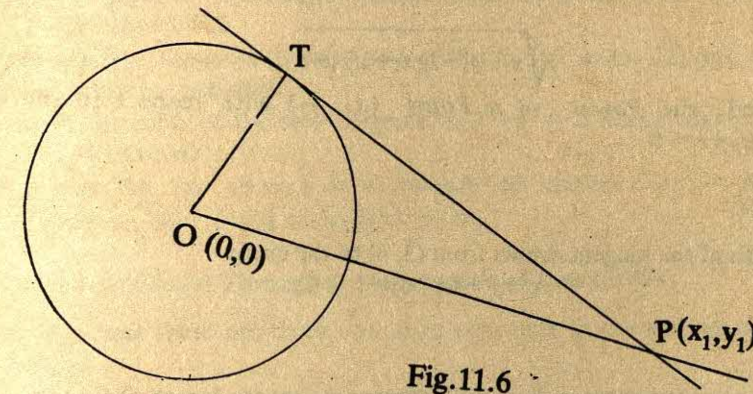


Fig.11.6

∴ Applying Pythagoras Theorem, we have

$$PT^2 = OP^2 - r^2$$

or
$$PT^2 = (x_1-0)^2 + (y_1-0)^2 - r^2$$
$$= x_1^2+y_1^2-r^2$$
$$\therefore\ PT = \sqrt{x_1^2+y_1^2-r^2}$$

If the length of the tangent is desired from the point (x_1, y_1) to the circle given by the general equation

$$x^2+y^2+2gx+2fy+c=0,$$

we can proceed in the similar way as in above case. The centre of the circle is $(-g, -f)$ and radius is $r=\sqrt{g^2+f^2-c}$. Therefore, applying again Pythagoras Theorem, we get

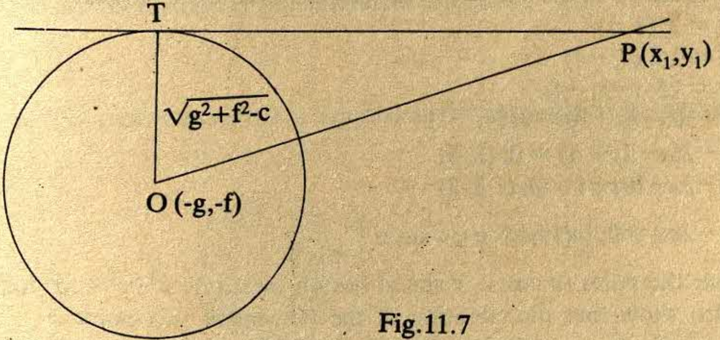


Fig.11.7

$$PT^2 = (x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$
$$= x_1^2+g^2+2gx_1+y_1^2+f^2+2y_1f-(g^2+f^2-c)$$
$$= x_1^2+y_1^2+2gx_1+2fy_1+c$$

Hence, the length of the tangent *PT* is given by

$$\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

PT^2 is called the *Power of a Point* (x_1, y_1) with respect to the circ $x^2+y^2+2gx+2fy+c=0$.

Example 11.14

Find the length of the tangent drawn from (3, 4) to the circle

$$x^2+y^2-4x+6y-1=0.$$

Solution

Substituting the coordinates in the given equation we get the length of the tangent

$$= \sqrt{3^2+4^2-4\times3+6\times4-1}$$

$$= \sqrt{9+16-12+24-1}$$

$$= \sqrt{36} = 6$$

Hence, the length of the required tangent is 6.

EXERCISE 11.6

1. Find the equation of the tangent to the given circles at the indicated points:
   - (i) $x^2+y^2=2$; $(1, 1)$
   - (ii) $x^2+y^2=10$; $(1, -3)$
   - (iii) $x^2+y^2=25$; $(-3, -4)$
   - (iv) $x^2+y^2-30x+6y+109=0$; $(4, -1)$
   - (v) $x^2+y^2-26x-2y+45=0$; $(2, 3)$
   - (vi) $x^2+y^2-2x-10y+1=0$; $(-3, 2)$
   - (vii) $x^2+y^2-2ax=0$; $\left\{a(1+\cos\alpha), a\sin\alpha\right\}$
2. Show that the point $(a\cos\alpha, a\sin\alpha)$ lies on the circle $x^2+y^2=a^2$ for all valu of α. Also show that the equation of the tangent at this point is $x\cos\alpha + y\sin\alpha = a$. Further show that the line $x\sin\alpha - y\cos\alpha = a$ is tangent to the circle for all values of α.
3. Find the condition that the line $lx+my=n$ is a tangent to the circle $x^2+y^2=a^2$.
4. Find the equations of the tangent lines to the circle $x^2+y^2=9$ which are paral to the line $2x+y-3=0$.
5. Find the length of the tangent drawn from the point $(-2, 3)$ to the cir $2x^2+2y^2=3$.

6. Find the length of the tangent drawn from the point (−2, −3) to the circle $x^2+y^2-2x-10y+1=0$.
7. Prove that the tangents to the circle $x^2+y^2=169$ at (5, 12) and (12, −5) are perpendicular to each other.
8. Find the equation of the two tangents to $x^2+y^2=3$ which makes an angle of 60° with the axis of x.
9. Show that the line $x+y\sqrt{3}=4$ touches the circles $x^2+y^2=4$ and $x^2+y^2-4x-4\sqrt{3}y+12=0$ at the same point.

11.7 Families of Circles Through the Intersection of Two Circles

We have seen that there are three unknown constants in the general equation of a circle given by

$$x^2+y^2+2gx+2fy+c=0$$

Thus, we require three conditions to determine a circle. If we impose two conditions on a circle, one of the constants in its equation will be undetermined. This condition may be chosen arbitrarily, provided the condition is independent of and does not contradict the conditions which determine the other two. We have, therefore, a system in which only one arbitrary constant appears; consequently, we have a one-parameter family of circles. Thus a one-parameter family of circles is formed in the same way as the imposition of one condition on a straight line leads to a one parameter family of straight lines.

The following examples will illustrate what we have said in the above development.

Example 11.15

Write the equation of the family of circles with centre at (−4, 3).

Solution

Since having the centre specified is equivalent to imposing two conditions on the

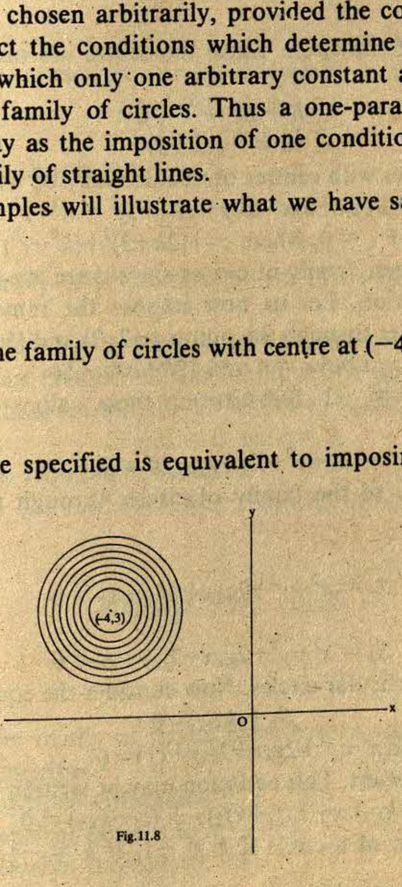


Fig. 11.8

circle, we can determine two constants, namely, h and k in the general equation the circle given by $x^2+y^2+2gx+2fy+c=0$.
Also since $g=4$ and $f=-3$, from the above equation, we have

$$x^2+y^2+8x-6y+c=0$$

In this equation c is the undetermined constant, that is, a parameter. We thus a one-parameter family of circles. Fig. 11.8 shows some members of this family various values of c.

A family of circles with the same centre and different radii is called a *family* *Concentric Circles*. The family of circles in the above example is a Family Concentric Circles.

Example 11.16

Find an equation of the circle which passes through the points $(-2, 2)$ and (5, and that has its centre on the line $x+2y+3=0$.

Solution

Let (h, k) be the centre of the circle. Since the centre lies on the line $x+2y+3=$ we have

$$h+2k+3=0$$

or

$$h=-(2k+3)$$

Then the family of all circles with centres on the line may be written as

$$x^2+y^2+2(2k+3)x-2ky+(2k+3)^2+k^2-r^2=0$$

or, $x^2+y^2+2(2k+3)x-2ky+c=0$, where $c=(2k+3)^2+(k^2-r^2)$ ...(11.21

Thus, we get a two-parameter family of circles since there are two arbitrary constant k and c left in the equation. Let us now impose the remaining two conditions namely, that the circle passes through the points $(-2, 2)$ and $(5, 1)$. We get

$$-12k+c=4 \text{ and } 18k+c=-56$$

Thus $c=-20$ and $k=-2$. Substituting these values in (11.21), we get the equation of the circle as

$$x^2+y^2-2x+4y-20=0.$$

Now we come actually to the family of circles through the intersection of tw circles.
Let

$$S_1=x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad \ldots(11.22$$

and

$$S_2=x^2+y^2+2g_2x+2f_2y+c_2=0 \qquad \ldots(11.23$$

be the equations of two particular circles. Now consider the equation

$$S_1+kS_2=0$$

i.e. $x^2+y^2+2g_1x+2f_1y+c_1+k(x^2+y^2+2g_2x+2f_2y+c_2)=0$ ...(11.24

where k is an arbitrary constant. This equation may be written in the form

$(1+k)x^2+(1+k)y^2+2(g_1+kg_2)\,x+2\,(f_1+kf_2)y+(c_1+kc_2)=0$ ...(11.25

This is clearly the equation of a circle if $k \neq -1$. If $k=-1$, (11.25) reduces to the equation of a straight line given by

$$S_1 - S_2 = 0$$

i.e. $2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_1)=0$

In either case, if the two circles have a point of intersection, say, (x_1, y_1) the curve given by (11.24) will pass through this point. This is so because if (x_1, y_1) satisfies the equations of the two circles, then substitutions of these coordinates in (11.24) reduces it to

$$0 + k\,(0) = 0.$$

We say that (11.24) or (11.25), for $k \neq -1$, represents a one-parameter family of circles, all members of which pass through the point of intersection of two given circles, if they have any. If $k = -1$, (11.24) and (11.25) reduce to the equation of a straight line which passes through the intersection of the two given circles, if they have any.

Examples 11.17

Find the equation of the circle through the intersection of the two circles

$$x^2+y^2-8x-2y+7=0$$
$$x^2+y^2-4x+10y+8=0$$

and that passes through the point $(-1, -2)$.

Solution

The equation of the family of circles through the intersection of the two given circles is

$$(x^2+y^2-8x-2y+7) + k\,(x^2+y^2-4x+10y+8) = 0$$

Since the circle is one member of this family and it passes through the point $(-1, -2)$, substituting these coordinates in the above equation, we get

$$24-3k=0 \text{ or, } k=8.$$

This value of k gives the required equation as

$$9x^2+9y^2-40x+78y+71=0.$$

EXERCISE 11.7

Find the equation of the family of circles described in each of the Exercises 1 to 3, and select the specified member.

1. Centre at $(2, -1)$; member with radius 3.
2. Centre at $(-4, 2)$; member tangent to $x-y=3$.
3. Centre at $(3, -5)$; member through $(0, -1)$.
4. Find the equation of the circle through the intersection of the two circles

$$x^2+y^2+8x-2y+7=0$$

and $$x^2+y^2-4x+10y+8=0$$

and which satisfies the following additional conditions:

(i) has its centre on the x-axis, and

(ii) passes through the point $(3, -3)$.

6. Find the equations of the circles which have centres on $2x-3y+4=0$.
7. Show that the equation of the family of circles which touch both the coordina axes can be put in the form $x^2+y^2-2hx-2hy+h^2=0$.

11.8. Condition for Two Intersecting Circles to be Orthogonal

Before we find the condition for two circles to be orthogonal, we define the ang between two curves and their orthogonality.

Definition

The angles between the tangents to the two curves intersecting at a point is calle the angle between the two curves intersecting at that point.

Definition 11.4

Two curves are said to intersect orthogonally when the tangents at the commo point are at right angles.

Now we find the condition of orthogonality.

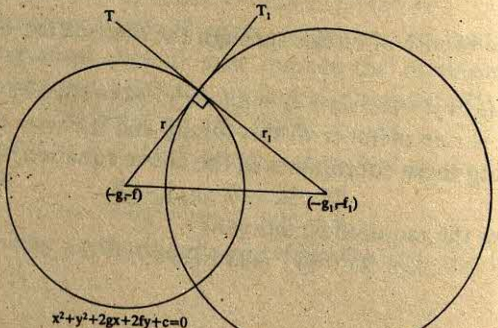


Fig.11.9

Let $$x^2+y^2+2gx+2fy+c=0 \qquad \text{... (11.2}$$

and $$x^2+y^2+2g_1x+2f_1y+c_1=0 \qquad \text{... (11.2}$$

be the equations of two given circles. The centres of the circles are $(-g, -f)$ an $(-g_1, -f_1)$ and their respective radii are $\sqrt{g^2+f^2-c}$ and $\sqrt{g_1^2+f_1^2-c_1}$.

Let PT and PT_1 be the tangents to the circles (11.26) and (11.27) respectively at the intersecting point P. Now, since as we know that the radius of a circle through a point is perpendicular to the tangent at that point, it easily follows that the angle of intersection of the circles is same as the angle between their radii drawn through the points of intersection. In the case of orthogonal intersection, this angle is $\frac{\pi}{2}$ and consequently the square of the distance between the centres of the circles is equal to the sum of the squares of their radii.

For orthogonal intersection we then have

$$(g-g_1)^2+(f-f_1)^2 = g^2+f^2-c+g_1^2+f_1^2-c_1$$

or $g^2+g_1^2-2gg_1+f^2+f_1^2-2ff_1 = g^2+f^2-c+g_1^2+f_1^2-c_1$

or

$$2gg_1+2ff_1 = c_1+c_2 \qquad (11.28)$$

This is the required condition.

Example 11.18

Find the equation of the circle which cuts orthogonally the circle $x^2+y^2-6x+4y-3=0$, passes through (3, 0) and touches the axis of y.

Solution

The equation of the circle which touches the axis of y is

$$x^2+y^2-2hx-2ky+k^2=0.$$

If it passes through (3, 0), then we have

$$k^2 = 6h-9 \qquad \text{... (11.29)}$$

If it cuts the given circle orthogonally,

$$6h-4k = k^2-3 \qquad \text{... (11.30)}$$

From (11.29) and (11.30), $k=3$ and $h=3$.

Therefore, the required equation is

$$x^2+y^2-6y-6y+9=0.$$

EXERCISE 11.8

1. Prove that the two circles which pass through two points $(0, a)$ and $(0, -a)$ and touch the straight line $y=mx+c$ will cut orthogonally if
$$c^2 = a^2(2+m^2)$$
2. Find the equation to the circle which passes through (1, 1) and cuts orthogonally each of the circles $x^2+y^2-8x-2y+16=0$ and $x^2+y^2-4x-4y-1=0$.
3. Let the equation of two circles whose radii are r, r' be $S=0$; $S'=0$; then show that the circles
$$\frac{S}{r} \pm \frac{S'}{r'} = 0$$
will intersect at right angles.

CHAPTER 12

CONIC SECTIONS

12.1 Introduction: Sections of a Cone

Consider a circle C. Let A be a line through the centre of C perpendicular to the plane of C, and let V be a point on A not in the plane of C (Fig. 12.1). Through a point P on C draw an infinite line passing through V. As P moves around C, the line PV sweeps out *right circular cone* with *axis A* and *vertex V*. Each of the lines PV is called a *generator* of the cone, and the angle α between the axes and any generator is called the *vertex angle*. As shown in the figure, the upper and lower portions of the cone that meet at the vertex are called *nappes* of the cone.

In the present context the cone is taken to be the surface we have just described, i.e. it consists of both endless *nappes* and not the solid cone.

The curves obtained by slicing a cone with a plane that does not pass through the vertex are called *Conic Sections* or simply *Conics*. If the slicing plane is parallel to a generator, the cone is called a *Parabola*. Otherwise, the cone is called an *Ellipse* or a *Hyperbola* depending on whether the plane cuts just one or both nappes. The hyperbola is to be thought of just a single curve consisting of two branches, one on each nappe. These curves are shown in Fig. 12.2.

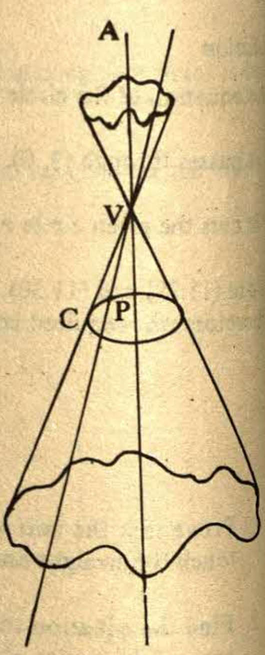
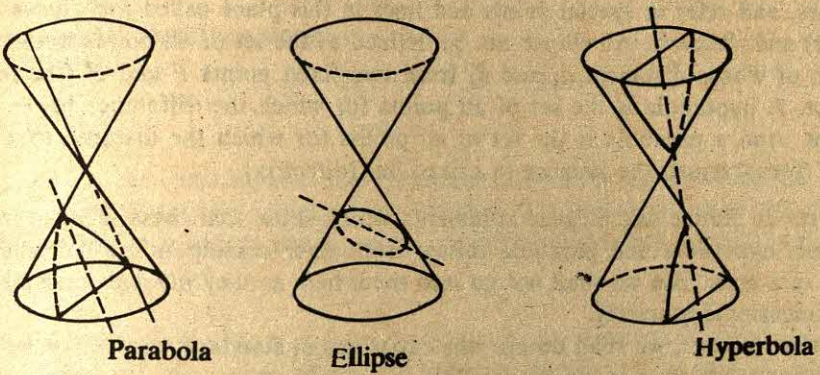


Fig. 12.1

It should be noted that if we move each of the slicing plane (Fig. 12.2) parallel to itself until it passes through the vertex, then the possibilities are that we get three so-called "degenerate" conics, namely a single straight line, a point, and a pair of intersecting straight lines respectively corresponding to parabola, ellipse and hyperbola.



Parabola (i)

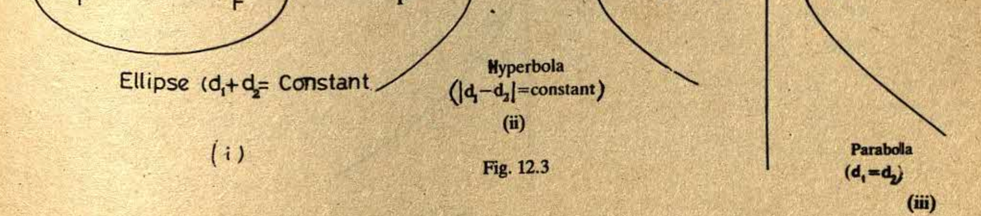
Ellipse (ii)

Hyperbola (iii)

Fig. 12.2

Many important discoveries, both in mathematics and science, have been linked to the conic sections. The greeks, particularly Archimedes and Apolonius studied these curves for their own beauty. The first application appeared almost 2000 years later when, about the year 1604, Galileo discovered that if a projectile is fired horizontally from the top of a tower, then its path would be a parabola. Only a few years later, in 1609, Kepler discovered that the orbit of Mars is an ellipse and went on to suggest that all planets move in elliptic paths. These and other developments took place hundreds of years ago, but the study of conic sections is far from outdated even today. Indeed these curves are important tools for present-day exploration of outer space and also for research into the behaviour of atomic particles.

We shall be studying the conic sections as plane curves. For this purpose it is convenient to use equivalent definitions that refer only to the plane in which the



Ellipse ($d_1+d_2=$ Constant

(i)

Hyperbola

($|d_1-d_2|$=constant)

(ii)

Fig. 12.3

Parabola

($d_1=d_2$)

(iii)

curve lies, and refer to special points and lines in this plane called *foci* (focus is the singular) and *directrix*. An ellipse can be defined as the set of all points in the plane the sum of whose distances d_1 and d_2 from two fixed points F and F' (the foci) is constant. A hyperbola is the set of all points for which the difference $|d_1 - d_2|$ is constant. And a parabola is the set of all points for which the distance to a fixed point F (focus) equals the distance to a fixed line (directrix).

There are simple and elegant arguments which show that these focal properties of ellipse, hyperbola and parabola follow from their corresponding definitions as section of a cone. But we shall not go into them here as they involve concepts from three dimensional geometry.

In what follows, we shall obtain the equations in standard forms of these conic sections and study them in greater detail.

12.2 Parabola

Definition 12.1

A *parabola* is the *locus* of all points whose distances from a fixed point equal their distances from a fixed line. The fixed point is called the *focus* and the fixed line the *directrix*.

To find the equation of a parabola we set up a coordinate system and place the directrix L and the focus F at convenient places. We place the focus at $(a, 0)$ $(a > 0)$ and let the directrix L be $x = -a$ as shown in Fig. 12.4. If $P(x, y)$ is a typical point on the parabola, then by using the distance formula the defining condition can be written as:

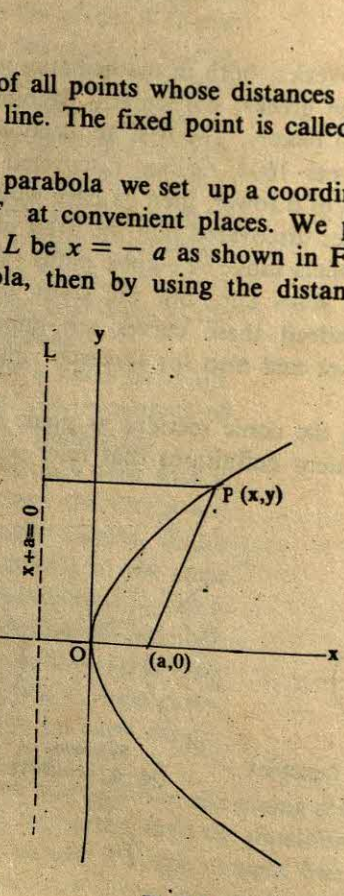


Fig.12.4

$$(x-a)^2+y^2=(x+a)^2$$

By squaring and simplifying, we get

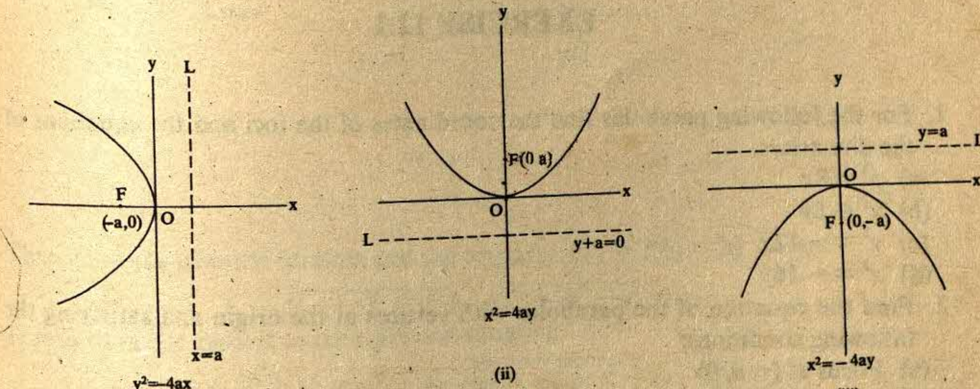
$$x^2+y^2-2ax+a^2=x^2+2ax+a^2$$

or $$y^2=4ax \qquad a>0 \qquad \text{...(12.1)}$$

Each point on the locus satisfies this equation. Conversely, by reversing the steps it can be shown that every point which satisfies this equation fulfils the defining condition. This is, therefore, the equation of the parabola with focus at $(a, 0)$ and directrix $x=-a$, $a>0$.

The line through the focus and perpendicular to the directrix is called the *axis* of the parabola and the point where the parbola intersects the axis is called the *vertex*. For the parabola (12.1), the axis is clearly the x-axis and the vertex if at the origin as is easily verified. If we replace y by $-y$, the equation (12.1) remains unchanged. Hence, the x-axis is the line of symmetry for this parabola. In fact the axis of the parabola is its line of symmetry.

If we change the position of the parabola relative to the coordinate axes, we naturally change its equation. Three other simple positions, each with its corresponding equation, are shown in Fig. 12.5. We shall call them the second, third and fourth standard forms of parabola. Students should verify the correctness of all three equations. It should be emphasised that the constant a is always understood to be positive. Geometrically, it is the distance from the focus to the vertex, and also from the vertex to the directrix.



(i) (ii) (iii)

Fig.12.5

Example 12.1

Show that the equation $y^2 - 8y - x + 19 = 0$ represents a parabola. Find its vertex, focus and directrix.

Solution

Completing the square on y, we get

$$(y-4)^2 = x - 3$$

If we introduce new variables $x' = x - 3$, $y' = y - 4$, the equation becomes $y'^2 = x'$ $\left(\text{which can be written as } y'^2 = 4.\left(\frac{1}{4}\right)x'\right)$. Thus in the new coordinate system, the vertex is at origin, focus at $\left(\frac{1}{4}, 0\right)$ and directrix $x' + \frac{1}{4} = 0$.

Hence, in the original coordinate system, the vertex is at (3, 4), focus at $\left(\frac{13}{4}, 4\right)$ and directrix $x = \frac{11}{4}$.

Note: In exactly the same way we see that any equation of the form $y^2 + Ax + By + C = 0$, $A \neq 0$ represents a parabola in the standard form I or II. Similarly, $x^2 + Ax + By + C = 0$, $B \neq 0$ will represent a parabola in standard form III or IV.

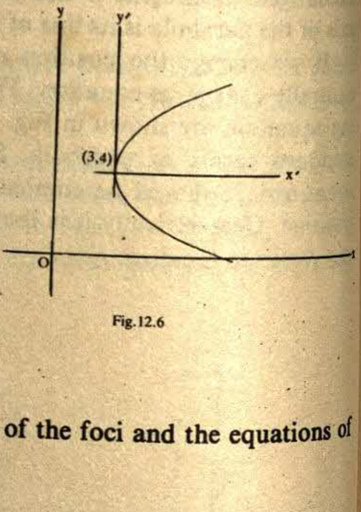


Fig.12.6

EXERCISE 12.1

1. For the following parabolas find the coordinates of the foci and the equations of the directrices.
   (a) $y^2 = 8x$
   (b) $x^2 = 6y$
   (c) $y^2 = -12x$
   (d) $x^2 = -16y$
2. Find the equation of the parabolas with vertices at the origin and satisfying the following conditions:
   (a) Focus at $(-a, 0)$
   (b) Directrix $y = 2$
   (c) Passing through (2, 3) and axis along x-axis
3. Find the foci, vertices, directrices and axes of the following parabolas. Also draw their rough sketches.

(a) $y = x^2 - 2x + 3$
(b) $y = -4x^2 + 3x$
(c) $x^2 + 2y - 3x + 5 = 0$

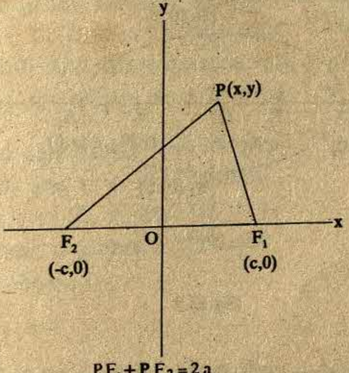
4. Find the equation of the following parabolas:
   (a) Directrix $x = 0$, focus at $(6, 0)$
   (b) Vertex at $(0, 4)$, focus at $(0, 2)$
   (c) Focus at $(-1, -2)$, directrix $x - 2y + 3 = 0$:

12.3 Ellipse

Definition 12.2

An *ellipse* is the locus of all points the sum of whose distances from two fixed points is constant. The two fixed points are called the foci (singular *focus*).

To find the equation of an ellipse in the standard form we choose a coordinate system and place the foci F_1 F_2 at convenient places. Let the distance between the foci be $2c$, and for any arbitrary point $P(x, y)$ on the ellipse let $PF_1 + PF_2 = 2a$ (constant). *For the definition to make sense* we must have $a > c$. Take the focus F_1 at $(c, 0)$ and F_2 at $(-c, 0)$ as in Fig. 12.7



$PF_1 + PF_2 = 2a$

Fig. 12.7

Then using the distance formula and the property $PF_1 + PF_2 = 2a$ we get

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

Transferring one radical to the right and squaring

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

Simplifying, we get

$$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x$$

Squaring again, $x^2 - 2cx + c^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2 = 0$

Simplifying, we get $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$

Since $a > c$, we can introduce a new quantity

$b = \sqrt{a^2 - c^2}$

and then we can write

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(12.2}$$

This is the *equation of an ellipse* in its standard form.

We have shown that every point on the ellipse satisfies this equation. Conversely, by reversing the steps, it can be shown that each point which satisfies this equatio is on the ellipse.

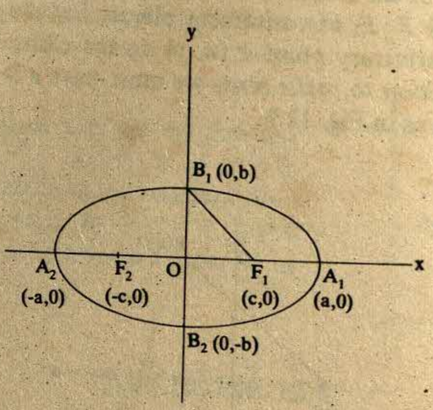
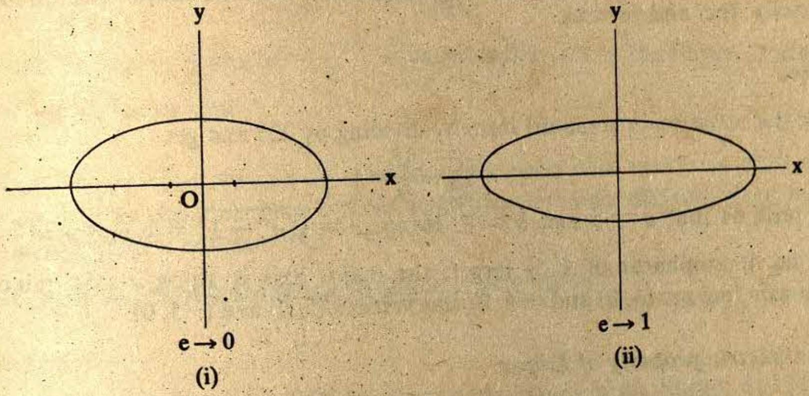


Fig.12.8

A sketch of such an equation is shown in Figure 12.8. From the equation, we see immediately that the curve is symmetric with respect to both x-and y-axes. The segment A_1A_2 is called the *major axis* and the segment B_1B_2 is called the *minor axis* of the ellipse, and the point O where these axes intersect is called the *centre*. The two points A_1 and A_2 at the ends of major axis are called the *vertices* of the ellipse. We see from Fig. 12.8 that since $B_1F_1 + B_1F_2 = 2a$ and $B_1F_1 = B_1F_2$, we get $B_1F_1 = a$ and hence $OB_1 = \sqrt{a^2 - c^2} = b$ or $B_1B_2 = 2b$. Thus, the *length of the minor axis* is $2b$. Again, since $A_1F_1 = A_2F_2$, we get $A_1A_2 = A_1F_1 + F_1A_2 = A_1F_1 + A_1F_2 = 2a$ (which is obvious otherwise from the figure also as A_1 is $(a, 0)$ and A_2 is $(-a, 0)$. Thus the *length of the major axis* is $2a$. The numbers a and b are called the *lengths of semi- major axis* and *semi-minor axix*.

The eccentricity e of the ellipse is defined as $e = \frac{c}{a} = \left(\sqrt{\frac{a^2-b^2}{a^2}}\right)$. Since $\sqrt{a^2-b^2} < a$, we have $0 < e < 1$.



(i)

(ii)

Fig.12.9

The eccentricity measures the flatness of the ellipse. If a is kept fixed and c is very small, then e is close to zero. This means the foci are close to each other. b is nearly as large as a, i. e. the ellipse is almost a circle as in Fig. 12.9 (i). In other words, a circle of radius a is the limiting position of the ellipse when the two foci actually coincide, i.e. $c = 0$ or $b = a$. On the other hand, if c is close to a, then e is near 1 (i.e. b is very small) and the ellipse is quite flat as shown in Fig. 12.9.(ii). The limiting position of the ellipse, as e becomes 1, is line segment of length $2a$.

In finding the standard equation of the ellipse, we placed the foci at $(c, 0)$ and $(-c, 0)$. If instead we put them on the y-axis at $(0, c)$ and $(0, -c)$ and carry through the same argument, then, the equation of the ellipse, we obtain, is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

The y-axis is now the major axis, the x-axis the minor axis, and The vertices are $(0, a)$ and $(0, -a)$. The quantity $b = \sqrt{a^2 - c^2}$ and the eccentricity $e = \frac{c}{a}$ are as before. Fig. 12.10 shows the sketch of this ellipse. Thus, in the standard form of the ellipse the foci, vertices and major axis are along the x-axis or y-axis depending on whether the denominator of x^2 or y^2 is larger in the standard form.

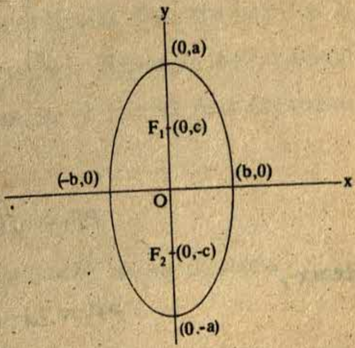


Fig.12.10

Example 12.2

Given the ellipse with equation $9x^2 + 25y^2 = 225$, find the major and minor axes, eccentricity, foci and vertices.

Solution

We put the equation in standard form by dividing by 225 and get

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

which tells us that $a = 5$ and $b = 3$. Hence, $c = \sqrt{a^2 - b^2} = 4$ and $e = \frac{c}{a} = \frac{4}{5}$. Since the denominator of x^2 is larger, the major axis is along x-axis, minor axis along y-axis, foci are $(4, 0)$ and $(-4, 0)$ and vertices $(5, 0)$ and $(-5, 0)$.

Focus-Directrix property of Ellipse

Consider the ellipse in its standard form, viz. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and draw two lines $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ as in Fig. 12.11. We know that

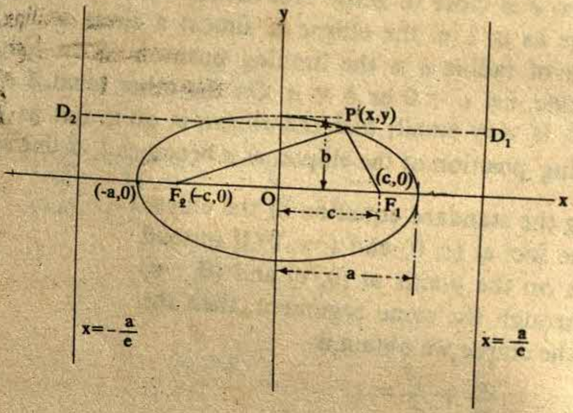


Fig.12.11

$$PF_1=\sqrt{(x-c)^2+y^2}=a-\frac{c}{a}x=e\left[\frac{a}{e}-x\right]$$

Hence,

$$PF_2=2a-PF_1=a+\frac{c}{a}x=e\left[x-\left(\frac{-a}{e}\right)\right]$$

The quantities in brackets can be interpreted as distances PD_1 and PD_2 from P to the lines $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ respectively. The above relations can, therefore, be written as

$$\frac{PF_1}{PD_1} = e \text{ and } \frac{PF_2}{PD_2} = e$$

Each of the lines $x = \frac{a}{e}$ and $x = \frac{-a}{e}$ is called a *directrix* of the ellipse. The above relation therefore shows that

"An Ellipse can be characterized as the locus of a point that moves in such a way that the ratio of its distance from a fixed point (focus) to its distance from a fixed line (directrix) equals a constant $e<1$."

This alternative way of characterizing an ellipse is very useful.

Example 12.3

Show that $4x^2 + 16y^2 - 24x - 32y = 12$ is the equation of an ellipse, and find its vertices, foci, eccentricity and directrices.

Solution

The equation can be written as

$$4(x-3)^2 + 16\,(y-1)^2 = 64$$

or

$$\frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1$$

Translating the coordinate axes so that the new coordinates are $X = x - 3$, $Y = y - 1$, the equation becomes $\frac{X^2}{16} + \frac{Y^2}{4} = 1$ which is an ellipse with $a = 4, b = 2$, $c = \sqrt{a^2 - b^2} = 2\sqrt{3}$, $e = \frac{c}{a} = \frac{\sqrt{3}}{2}$. In the new coordinates: foci are $(\pm 2\sqrt{3}, 0)$, vertices $(\pm 4, 0)$ and directrices $X = \frac{8}{\sqrt{3}}$, $X = -\frac{8}{\sqrt{3}}$. Hence, in the original coordinate system (x, y), foci are $(3 \pm 2\sqrt{3}, 1)$, vertices are $(3 \pm 4, 1)$, and directrices $x = 3 + \frac{8}{\sqrt{3}}$ and $x = 3 - \frac{8}{\sqrt{3}}$.

Example 12.4

Find the equation of the ellipse with foci at $(\pm 5, 0)$ and $x = \frac{36}{5}$ as one directrix.

Solution

We have $c = 5, \frac{a}{e} = \frac{36}{5}$. Since $e = \frac{c}{a}$ we get $\frac{a^2}{c} = \frac{36}{5}$ i.e. $a = 6$.

Then $b = \sqrt{a^2 - c^2} = 11$. Thus the equation is

$$\frac{x^2}{36} + \frac{y^2}{11} = 1.$$

EXERCISE 12.2

1. For the following ellipses find the lengths of major and minor axes, coordinate of foci and vertices, and the eccentricity.
   (a) $16x^2 + 25y^2 = 400$
   (b) $9x^2 + 16y^2 = 144$
   (c) $3x^2 + 2y^2 = 6$
   (d) $x^2 + 4y^2 - 2x = 0$
2. Find the equation of the ellipse satisfying the given conditions.
   (a) Vertices at $(\pm 5, 0)$, foci at $(\pm 4, 0)$
   (b) Vertices at $(0, \pm 10)$, $e = \frac{4}{5}$
   (c) Foci at $(0, \pm 4)$, $e = \frac{4}{5}$
   (d) Axes along coordinate axes, passing through $(4, 3)$ and $(-1, 4)$
   (e) Foci at $(\pm 3, 0)$, passing through $(4, 1)$
   (f) $e = \frac{3}{4}$, foci on y-axis, centre at origin, passing through $(6, 4)$
3. Find the equation of the locus of all points the sum of whose distances from $(3, 0)$ and $(9, 0)$ is 12.
4. Find the equation of the locus of all points whose distances from $(0, 4)$ are $\frac{2}{3}$ of their distances from the line $y = 9$.

12.4 Hyperbola

Definition 12.3

A hyperbola is the *locus* of all points the difference of whose distances from two fixed points is a positive constant. The two fixed points are called *foci.*

To obtain the equation of a hyperbola in standard form we choose a coordinate system and place the foci F_1 and F_2 conveniently. Let the distance between the foci be $2c (c > 0)$.

We take $F_1 (c, 0)$ and F_2 at $(-c, 0)$. Let $P (x, y)$ be any point on the hyperbola Then the condition of the locus asserts

$|PF_1 - PF_2| = 2a \ (a > 0)$

It is obvious that the locus shall consist of two branches, for the right one $PF_1 - PF_2 = -a$ and for left one $PF_1 - PF_2 = +a$.

Using the distance formula we obtain

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a.$$

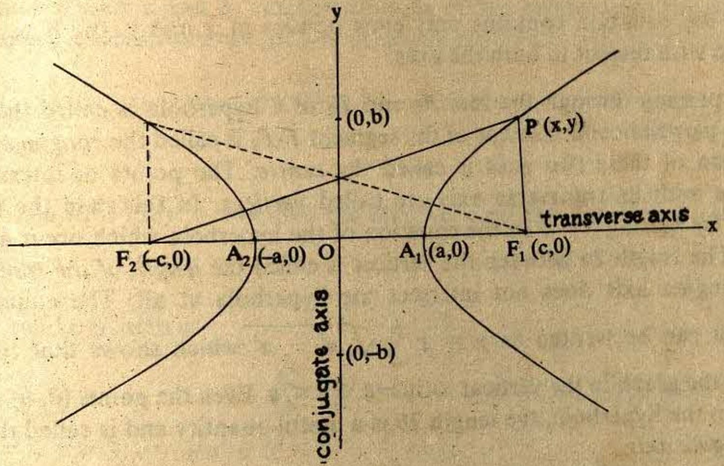


Fig.12. 12

Transferring one radical to the right and squaring

$$(x+c)^2 + y^2 = (x-c)^2 + y^2 \pm 4a\sqrt{(x-c)^2 + y^2} + 4a^2$$

Simplifying, we get

$$PF_1 = \sqrt{(x-c)^2 + y^2} = \pm\left(\frac{c}{a}x - a\right) \qquad \text{...(12.3)}$$

and

$$PF_2 = \sqrt{(x+c)^2 + y^2} = \pm\left(\frac{c}{a}x + a\right) \qquad \text{... (12.4)}$$

(the second equation is obtained by using $PF_2 - PF_1 = \pm 2a$.). Squaring the equation (12.3) once again, we get

$$x^2 - 2cx + c^2 + y^2 = \frac{c^2}{a^2}x^2 - 2cx + a^2$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

We observe that in the triangle PF_1F_2 (Fig. 12.12) $PF_2 < PF_1 + F_1F_2$ or $PF_2 - PF_1 < F_1 F_2$ i.e. $2a < 2c$ or $a < c$ and hence $c^2 - a^2$ is a positive number which we denote by b^2. Hence, $b^2 = c^2 - a^2$

Thus we can write the equation of the hyperbola in the standard form as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(12.5)}$$

Since the equation contains only even powers of x and y, the hyperbola too is symmetric with respect to both the axes.

The line passing through the foci F_1 and F_2 of a hyperbola is called the *transverse axis*. The perpendicular bisector of the segment F_1F_2 is called the *congjugate axis*. The intersection of these two axes is called the *centre*. The points of intersection of a hyperbola with its transverse axis are called *vertices*. In this case the vertices are obtained by putting $y = 0$ in the equation of the hyperbola which occur at $(a, 0)$ and $(-a, 0)$. The length $2a$ between the vertices is called the *length of the transverse axis*. The conjugate axis does not intersect the hyperbola at all. The equation of the hyperbola can be written as $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$ which shows that there are no points on the graph in the vertical strip $-a < x < a$. Even the points $(0, b)$ and $(0, -b)$ are not on the hyperbola, the length $2b$ is a useful quantity and is called the *length of the conjugate axis*.

The *ecentricity* of a hyperbola is defined as

$$e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} = \sqrt{1+\frac{b^2}{a^2}}$$

The eccentricity of a hyperbola is thus always greater than 1.
If the foci are placed at $(0, c)$ and $(0, -c)$, the equation takes the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where as before $b = \sqrt{c^2 - a^2}$, and the curve is shown in Figure 12.13 where the transverse axis is now along y-axis and conjugate axis along x-axis.

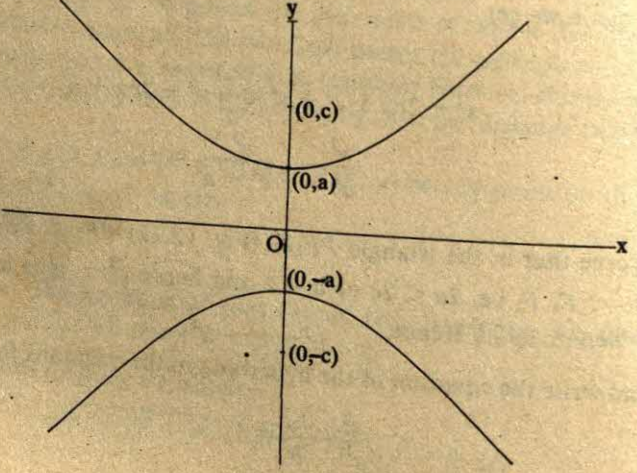


Fig. 12.13

Note that the relative size of *a* and *b* does not matter here as it was in the case of an ellipse. The numbers *a* and *b* can, therefore, be of any relative size. In particular, they can be equal, in which case we get a *rectangular hyperbola* with equation

$$x^2 - y^2 = a^2 \text{ or } y^2 - x^2 = a^2$$

Example 12.5

For the hyperbola $9x^2 - 16y^2 = 144$, find the axes, vertices, foci and eccentricity.

Solution

The equation can be written as

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Therefore, $a = 4, b = 3, c = \sqrt{a^2 + b^2} = 5$
The transverse and conjugate axes are along x and y-axes respectively, vertices are at ($\pm$ 4, 0) and foci at ($\pm$ 5, 0). The eccentricity $e = \frac{c}{a} = \frac{5}{4}$.

Example 12.6

Find the equation of the hyperbola with vertices at (0, $\pm$ 6) and $e = \frac{5}{3}$. Locate its foci.

Solution

Since the vertices are along the y-axis, the equation is of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

with $a = 6, e = \frac{c}{a}$ or $c = a.e = \frac{6 \times 5}{3} = 10$

Therefore, $b = \sqrt{c^2 - a^2} = \sqrt{100 - 36} = 8$.

Hence, the equation is $\frac{x^2}{36} - \frac{y^2}{64} = 1$
and the foci are at (0, $\pm$ 10).

Focus-Directrix Property of Hyperbola

Using the eccentricity $e = \frac{c}{a}$, the equation giving the lengths PF_1 and PF_2 can be written as

$$PF_1 = \pm (ex - a) = \pm e\left[x - \frac{a}{e}\right] \qquad \text{...(12.6)}$$

$$PF_2 = \pm (ex + a) = \pm e\left[x - \left(-\frac{a}{e}\right)\right] \qquad \text{...(12.7)}$$

where the plus sign apply to the right branch of the curve and minus sign to the left branch (See Fig. 12.14)). If P lies on the right branch, as shown in the figure, then the quantities in the brackets can be interpreted as the distances PD_1 and PD_2 from

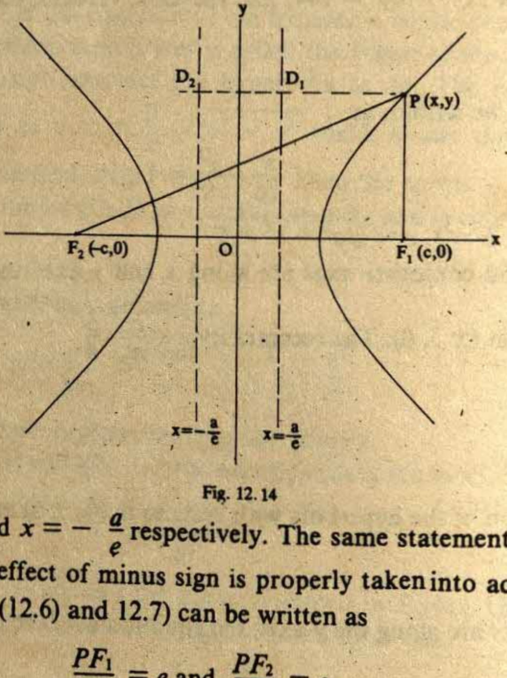


Fig. 12.14

P to the lines $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively. The same statement is true if P lies on the left branch, if the effect of minus sign is properly taken into account. Therefore in all cases the equations (12.6) and 12.7) can be written as

$$\frac{PF_1}{PD_1} = e \text{ and } \frac{PF_2}{PD_2} = e$$

Each of the line $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ is called a *directrix* of the hyperbola. We have thus shown that:

A hyperbola can be characterized as the locus of a point that moves in such a way that the ratio of its distance from a fixed point (a focus) to its distance from a fixed line (the corresponding directrix) equals a constant $e > 1$.

Just as in the the case of ellipse, this alternative definition of a hyperbola is very useful.

Example 12.7

Find the equation of the hyperbola whose vertices are (±6, 0) and one of the directrices is $x = 4$.

Solution

Let the equation be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Given that $a = 6$ and $\frac{a}{e} = 4$ i. e. $e = \frac{3}{2} = \sqrt{1 + \frac{b^2}{a^2}}$ which gives $b^2 = 45$. Hence, the required equation is

$$\frac{x^2}{36} - \frac{y^2}{45} = 1.$$

EXERCISE 12.3

1. For the following hyperbolas find the lengths of transverse and conjugate axes, eccentricity and the coordinates of foci and vertices:
   (a) $16x^2 - 9y^2 = 144$
   (b) $2x^2 - 3y^2 - 6 = 0$
   (c) $3x^2 - 2y^2 = 1$
2. Find the equations of the hyperbolas satisfying the following conditions:
   (a) Vertices $(\pm 5, 0)$, foci $(\pm 7, 0)$
   (b) Vertices $(0, \pm 7)$, $e = \frac{4}{3}$
   (c) Foci $(0, \pm 10)$, passing through $(2, 3)$
3. Find the equation of the locus of all points such that the difference of their distances from $(4, 0)$ and $(-4, 0)$ is always equal to 2.
4. Show that the equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents a hyperbola. Find the lengths of axes and eccentricity.

12.5 Condition for Tangency of the Line $y = mx + c$

In what follows, we shall study the intersection of the line $y = mx + c$ with the given conic. Since the equation of every conic is of second degree, we shall in general find two points of intersection. Again, since a tangent is a line which intersects the curve in two coincident points, we shall use this property to find the condition for tangency of the line $y = mx + c$ to the given conic.

(a) *Parabola:* Consider the intersection of the line $y = mx + c$ with the parabola

$y^2 = 4ax$, ... (12.8)

Substituting the value of y from the equation of the line in the equation of the parabola, we find that x-coordinate of the points of intersection of the line with the parabola are the roots of the equation

$$(mx + c)^2 = 4ax$$

i.e.
$$m^2x^2 + 2(mc - 2a)x + c^2 = 0 \qquad \text{...(12.9)}$$

This quadratic equation generally gives two values of x, except when $m = 0$ when one of the roots becomes infinite. Hence, the line intersects the parabola in general in two points except when it is parallel to the axis of the parabola in which case one of the points of intersection lies at infinity. If the line is to be a tangent to the parabola, the roots of the above quadratic should be coincident. The condition for this is

$$(mc - 2a)^2 = m^2c^2$$

i. e. $c = \frac{a}{m}$

The line

$$y = mx + \frac{a}{m}$$

therefore is tangent to the parabola $y^2 = 4ax$ for all values of m.

With this value of c, the two coincident roots of the quadratic (12.9) are $x' = \frac{a}{m^2}$. Substituting this in the equation of the parabola, we get $y' = \frac{2a}{m}$.

Hence, the point of contact of the tangent $y = mx + \frac{a}{m}$ with the parabola $y^2 = 4ax$ is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

It may be noted in passing that the points $(at^2, 2at)$ for all values of t, lies on the parabola $y^2 = 4ax$. We say that the parametric equation of the parabola is

$$x = at^2, y = 2at$$

The point $(at^2, 2at)$ is usually referred to as the point "t" on the parabola.

(b) *Ellipse:* Consider the intersection of the line $y = mx + c$ with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. As before, the x-coordinate of the point of intersection is given by

$$b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

or
$$(a^2m^2 + b^2)x^2 + 2mxa^2c + a^2(a^2 - b^2) = 0$$

which gives two roots of x, and hence two points of intersection. In order that the line may be a tangent, i.e. the two points of intersection may be coincident, we must have

$$m^2c^2a^4 = 4a^2(a^2 - b^2)(a^2m^2 + b^2)$$

or
$$c^2 = a^2m^2 + b^2$$

Hence, the line $y = mx + \sqrt{a^2m^2 + b^2}$ is a tangent to the ellipse for all values of m.

It may be noted that in fact we get two values of c with opposite signs, and hence the line $y = mx - \sqrt{a^2m^2 + b^2}$ is also a tangent to the ellipse. Thus we get two parallel tangents (as in the case of a circle).

As before, the point of contact of this tangent with the ellipse is

$$\left(\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \frac{b^2}{\sqrt{a^2m^2+b^2}} \right)$$

It is seen that the point $(a \cos\theta, b \sin\theta)$ lies on the ellipse for all values of θ. Hence, the parametric equation of the ellipse is

$$x = a \cos\theta, \ y = b \sin\theta$$

θ is called eccentric angle.

(c) *Hyperbola:* Considering the intersection of the line $y = mx + c$ with the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we find that the x-coordinate of the points of intersection is given by

$$b^2x^2 - a^2(mx+c)^2 - a^2b^2 = 0$$

or

$$(a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2 + c^2) = 0$$

For the line to be a tangent, i.e. the two roots of this quadratic equation to be coincident, we must have

$$c^2 = a^2m^2 - b^2$$

Hence, the line $y = mx + \sqrt{a^2m^2 - b^2}$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for all values of m.

Proceeding as before, we get the point of contact

$$\left(\frac{ma^2}{\sqrt{a^2m^2-b^2}}, \frac{b^2}{\sqrt{a^2m^2-b^2}} \right)$$

Here also we shall get another parallel tangent

$$y = mx - \sqrt{a^2m^2 - b^2}$$

It may also be noted that the point $(a \sec\theta, b \tan\theta)$ lies on the hyperbola for all values of θ, and hence the parametric equation of the hyperbola can be written as

$$x = a \sec\theta, \ y = b \tan\theta$$

Example 12.8

Find the equation to the chord of the parabola $y^2 = 4ax$ joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$. Hence deduce the equation to the tangent at $(at^2, 2at)$.

Solution

Required equation of the chord is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$= \frac{2}{t_1 + t_2}(x - at_1^2)$$

or

$$(t_1 + t_2)y = 2(x + at_1t_2)$$

If $t_1 \to t_2$ this becomes $t_1 y = x + at_1^2$

Hence the equation of tangent at $(at^2, 2at)$ is

$$ty = x + at^2$$

Example 12.9

Show that the line $x \cos\alpha + y \sin\alpha = p$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

Solution

We can write the equation of the line as $y = mx + c$ where $m = -\cot\alpha$, $c = p \operatorname{cosec}\alpha$. That is a tangent if

i.e.

$$c^2 = a^2m^2 + b^2$$

or

$$p^2 \operatorname{cosec}^2\alpha = a^2\cot^2\alpha + b^2$$

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

EXERCISE 12.4

1. Show that the equation of the tangent to the parabola $y^2 = 4ax$ which makes an angle θ with its axis is $y = x\tan\theta + a\cos\theta$.
2. Prove that the line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$ if $ln = am^2$.
3. Show that $y - x - 2 = 0$ is tangent to the parabola $y^2 = 8x$. What is the point of contact?
4. A line $lx + mx + n = 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that $a^2l^2 + b^2m^2 = n^2$.
5. Find the equation of the tangent at the point $(at^2, 2at)$ to the parabola $y^2 = 4ax$.
6. Find the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point $(a\cos\theta, b\sin\theta)$.
7. Find the equations of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the line $y = 3x + 7$. Also find the point of contact.
8. Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$ which is parallel to the line $4y = 5x + 7$.

CHAPTER 13

TRIGONOMETRIC FUNCTIONS

13.1. Introduction

Trigonometry means the science of measuring triangles. This science is more than two thousand years old. One of its uses lies in being able to determine heights and distances which are not easy to measure. This has been very useful in surveying navigation and astronomy. The methods of trigonometry and the properties of functions find their use in every branch of science and engineering. We study the properties of trigonometric functions and collect various formulas connecting them in chapter 13 The method of solving triangles and applications of trigonometry in determining heights and distances are discussed in chapter 14. Inverse trigonometric functions and their properties are studied in chapter 15.

13.2 Angles

An angle is considered as the figure obtained by rotating a given ray about its end-point. The original ray is called the *initial side* and the ray into which the initial side rotates is called the *terminal side* of the angle.

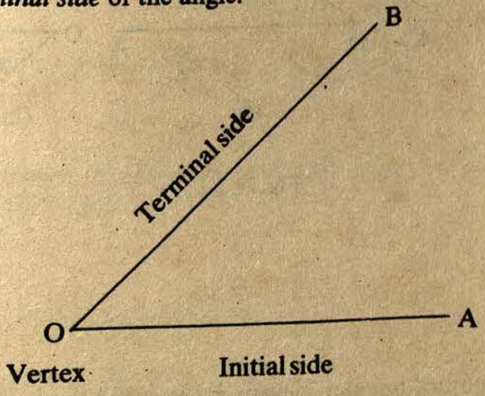


Fig.13.1

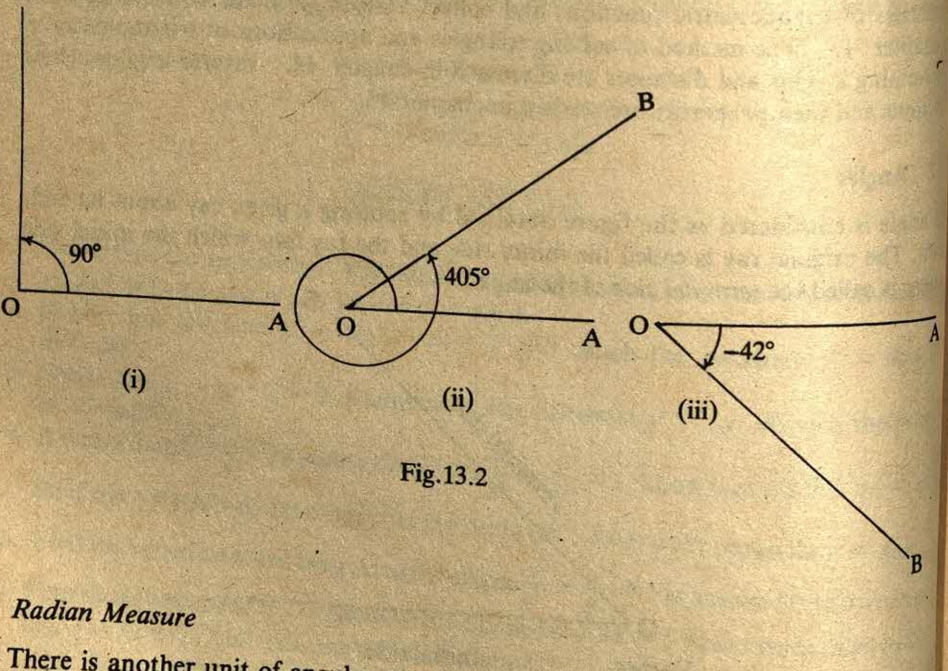
The *sense* of an angle is derived from the direction of rotation of the initial si into the terminal side. If this direction is counter-clockwise, the sense of the angle said to be positive and if this direction is clockwise then the sense is negative.

The measure of an angle is the amount of rotation required to get to th terminal side from the initial side. We place the vertex of the angle at the centre of circle of some fixed radius. Divide the circumference of the circle into 360 equa parts, called degrees. The number of degrees on the circumference between the initia and terminal sides of the angle is its degree measure. For additional precision, eac degree is subdivided into 60 equal parts, called minutes and each minute is divide into 60 equal parts called seconds. The symbol ° is used to denote degrees, ′ is use to denote minutes and ″ is used to denote seconds.

Remark

In our discussion in the above paragraph we considered a circle of fixed radius However, the measure of an angle does not depend on the radius of the circle.

Some of the angles are shown in the following figure. Note that the terminal side has to be rotated so as to have more than one revolution if the corresponding angl is greater then 360°.



(i) (ii) (iii)

Fig.13.2

Radian Measure

There is another unit of angular measurement called the *radian measure,* which is o particular importance in higher mathematics and its applications. This is based upo the fact that the ratio of the circumference of circle to its diameter is constant. This constant π is an irrational number having the non recurring decimal expansion $\pi = 3.14159\ldots$; $\frac{22}{7}$ is taken as an approximate value of π. As shown in the figure below, we place the vertex of the angle at the centre of the circle of radius r. If the length of the arc subtending the angle at the centre is s, the *radian measure* t of the angle is defined to be $\frac{s}{r}$. Note that the length of the arc is taken to be negative if we measure it in the clockwise direction. We again remark that the measure of an angle is independent of the radius of the circle considered because of the fact that the ratio of the circumference of a circle to its diameter is constant.

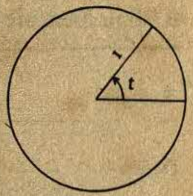


Fig.13.3

The radian measure t of the angle formed by one complete revolution of the initial side is $\frac{2\pi r}{r} = 2\pi$. Thus 2π radians $= 360°$ or π radians $= 180°$. Hence

$$1 \text{ radian} = \frac{180°}{\pi} = 57°\ 16' \text{ approximately}$$

and $1° = \frac{\pi}{180}$ radian $= 0.01746$ radian approximately.

Note that the definition $t = \frac{s}{r}$ can be used to find one of s, t and r provided the other two are given.

Example 13.1

Find the length of the arc of a circle of radius 5 cm subtending an angle measuring 45°.

Solution

$t = \frac{\pi}{180} \times 45 = \frac{\pi}{4}$. Hence $s = r \times t = \frac{5\pi}{4}$cm.

Example 13.2

Suppose arcs of same length in two circles subtend angles of 60° and 75° at the centre. Find the ratio of their radii.

Solution

Let r_1, r_2 denote the radii of the two circles. Now

$$60° = \frac{\pi}{180} \times 60 = \frac{\pi}{3} \text{ and } 75° = \frac{\pi}{180} \times 75 = \frac{5\pi}{12}.$$

Let s be the length of the arcs. Then

$s = \frac{\pi}{3} \cdot r_1 = \frac{5\pi}{12} \cdot r_2.$

Hence $r_1 : r_2 = 5 : 4$

Radian measure of some common angles are given in the following table·

Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Degrees	30°	45°	60°	90°	180°	270°	360°

Note that when an angle is expressed in radians, the word 'radian' is often omitted. Thus $\pi = 180°$ is really a short form of writing π radians $= 180°$.

EXERCISE 13.1

1. Find the radian measure corresponding to the following degree measures·
   (a) 15° (b) −22° 30′ (c) 340° (d) 420°
2. Find the degree measure corresponding to the following radian measures
   (a) $\frac{1}{4}$ (b) -2 (c) $\frac{7\pi}{3}$ (d) $\frac{5\pi}{6}$
3. Find the length of the arc of a circle of radius 5 cm subtending a central angle measuring 15°.
4. In a circle of diameter 40 cm. the length of a chord is 20 cm. Find the length of the minor arc of the chord.
5. A wheel makes 180 revolutions in one minute. Through how many radians does it turn in one second.
6. Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of 11 cm (use $\pi = \frac{22}{7}$).
7. Find the angle through which a pendulum swings if its length is 50 cm and the tip describes an arc of length (a) 10 cm (b) 16 cm (c) 26 cm.
8. Find the angle between the minute hand of a clock and the hour hand when the time is 7.20.

13.3 Circular Functions or Trigonometric Functions

Let θ be the angle XOC as shown in the figure and let $P(x, y)$ be any point other than O on its terminal side OC. Suppose $OP = r > 0$. We define the following functions known as circular or trigonometric functions. These are also called trigonometric ratios.

sine $\theta = \sin\theta = \frac{y}{r}$

cosine $\theta = \cos\theta = \frac{x}{r}$

tangent $\theta = \tan\theta = \frac{y}{x}$

cosecant $\theta = \operatorname{cosec}\theta = \frac{r}{y}$

secant $\theta = \sec\theta = \frac{r}{x}$

cotangent $\theta = \cot\theta = \frac{x}{y}$

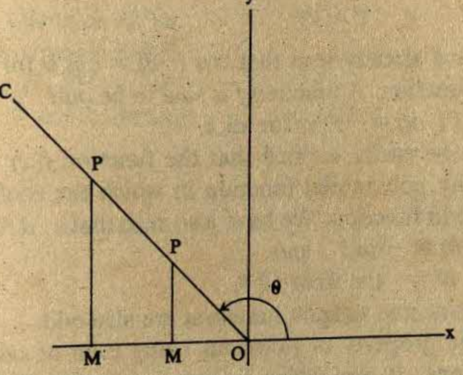


Fig.13.4.

We observe that the above functions depend only on the value of the angle θ and not on the point P chosen on the terminal side of the angle θ. For example, if we take another point $R'(x', y')$ on OC with $OP' = r'$ then considering similar triangles we obtain

$$\frac{y}{r} = \frac{y'}{r'},\ \frac{x}{r} = \frac{x'}{r'},\ \frac{y}{x} = \frac{y'}{x'}$$

This is also true if the terminal side coincides with one of the axes with the only difference that if it coincides with x-axis, then cosec and cot are not defined while in case it coincides with y-axis, then sec and tan are not defined. From the defintion and Fig. 13.5, it is clear that

$\sin\theta = \sin(\theta + 2\pi)$ $\quad \sin(-\theta) = -\sin\theta$

$\cos\theta = \cos(\theta + 2\pi)$ $\quad \cos(-\theta) = -\cos\theta$

$\tan\theta = \tan(\theta + 2\pi)$ $\quad \tan(-\theta) = -\tan\theta$

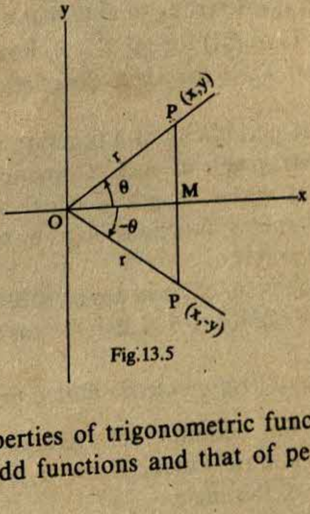


Fig.13.5

At this stage, let us notice two important properties of trigonometric functions. Let us briefly introduce the notion of even and odd functions and that of periodic functions.

A function f is said to be *even*
if $\quad f(x) = f(-x)$ for all x.

A simple example of an even function is a constant function.

Any polynomial function $p(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$ is an even function.

We have already seen that $\cos(-\theta) = \cos\theta$ for all θ. Thus cosine function is also an even function. A function f is said to be *odd*
if $\quad f(-x) = -f(x)$ for all x.
It can be easily verified that the function $f(x) = x$, $f(x) = x^3$ are odd functions. In fact any polynomial function in which the coefficients of even powers of x are zero is an odd function. We have also seen that
$\sin(-\theta) = -\sin\theta$ and
$\tan(-\theta) = -\tan\theta$ for all θ.
Thus sine and tangent functions are also odd.

The property of functions being even or odd is very useful in the study of such functions. It also helps in drawing graph of such functions as once we draw the graph for $x \geq 0$, we can complete the graph of f for all x easily.

The other important property of trigonometric functions which we want to observe now is that of periodicity. A function f is said to be *periodic* if there exists a real $T > 0$ such that $f(x+T) = f(x)$ for all x. We have already noted that
$\sin(\theta + 2\pi) = \sin\theta$ and
$\cos(\theta + 2\pi) = \cos\theta$.
Thus sine and cosine functions are examples of periodic functions. If a function f is periodic then the smallest $T > 0$, if it exists, such that

$$f(x+T) = f(x) \text{ for all } x$$

is called the period of the function. It can be easily seen that the period of the sine and cosine functions is 2π. We shall see later that the period of the tangent function is π. It is interesting to note that a constant function f is periodic as $f(x+T) = f(x)$ for all $T > 0$, however it does not have a period because there is no *smallest* $T > 0$ for which the relation holds.

The periodicity of a function is also very useful concept. In particular, it follows that the graph of such a function is completely known once we know it over an interval whose length is equal to the period of the function. The periodicity of trigonometric functions helps us to compute the values of these functions for large θ. For example,
$\sin 785^0 = \sin 65°$ and $\cos(-2070°)$
$= \cos(-2070° + 6 \times 360°) = \cos 90°$.

Values of Trigonometric Functions for the Angles 0°, 30°, 45°, 60°, 90°

From the definition of trigonometric functions it follows that the values of all trigonometric functions for a given angle are known, once we find the sine and cosine of the angle.

It is clear from the definition that
$\sin 0° = 0$, $\cos 0° = 1$, $\sin 90° = 1$ and $\cos 90° = 0$.
Let us calculate the values of these for 30°, 45° and 60°.
30°.

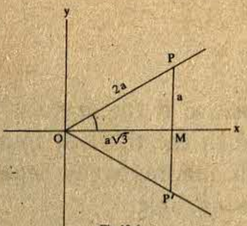


Fig.13.6

In Fig. 13.6, $PM = MP'$. The two triangles OPM and $OP'M$ are congruent. Hence, the triangle OPP' is an equilateral triangle. Therefore, if $PP' = 2a$, then $OP = 2a$, $PM = a$ and $OM = a\sqrt{3}$. Hence, $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.

45°:

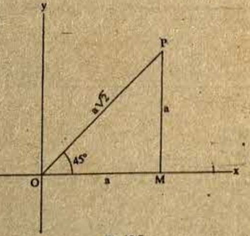


Fig.13.7

In Fig. 13.7, angles POM and OPM are equal. Hence, $OM = PM = a$ (say). Then $OP = a\sqrt{2}$ and

$\sin 45^0 = \frac{1}{\sqrt{2}}$ and $\cos 45^0 = \frac{1}{\sqrt{2}}$.

60°:

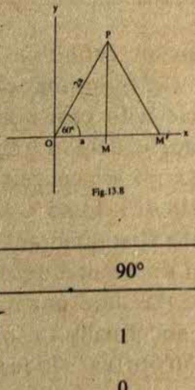


Fig.13.8

In Fig. 13.8, $OM = MM'$, the triangles OPM and MPM' are congruent and the triangle OPM' is equilateral. Hence, if $OM = a$, then $OP = 2a$ and $PM = a\sqrt{3}$. Therefore, $\sin 60° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$

Thus we have the following table:

θ	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Notational Convention

Since angles are measured either in radians or degrees, we adopt the convention that whenever we write $\cos \theta°$, we mean the cosine of the angle whose degree measure is $\theta°$ (and similarly for other ratios) and whenever we write $\cos \beta$, we mean the cosine of the angle whose measure in radians is β. If no such convention used, it should be clear from the context, which meaning is being used.

Signs of the Trigonometric Functions

The signs of the trigonometric functions depend on the quadrant in which the angle terminates. Thus $\sin \theta = \frac{y}{r}$ has the sign of y as r is always positive. Therefore $\sin\theta$

as positive if the *angle* is in first or second quadrant, while it is negative for the *angles* in the third or fourth quadrant. Similarly cos θ is positive in the first and fourth quadrants and negative in the remaining two quadrants. In fact we have the following table:

	I	II	III	IV
sin θ	+	+	−	−
cos θ	+	−	−	+
tan θ	+	−	+	−
cosec θ	+	+	−	−
sec θ	+	−	−	+
cot θ	+	−	+	−

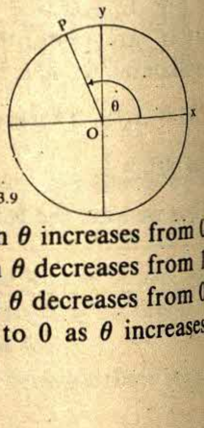


Fig.13.9

From the definition of trigonometric functions we observe the following useful facts. If P is the point on the circle with centre at origin O and radius 1 unit and if angle POX is θ then $\sin\theta = y$ and $\cos\theta = x$, where (x, y) are coordinates of P. It is also clear that $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$ for all θ.

In the first quadrant as the angle increases from 0° to 90°, sin θ increases from 0 to 1. In the second quadrant as θ increases from 90° to 180°, sin θ decreases from 1 to 0. In the third quadrant as θ increases from 180° to 270°, sin θ decreases from 0 to -1 and finally in fourth quadrant sin θ increases from -1 to 0 as θ increases from 270° to 360^0. In fact we have the following table:

	II *quadrant*		I *quadrant*
sine	decreases from 1 to 0	sine	increases from 0 to 1
cosine	decreases from 0 to -1	cosine	decreases from 1 to 0
tangent	increases from $-\infty$ to 0	tangent	increases from 0 to ∞
cotangent	decreases from 0 to $-\infty$	cotangent	decreases from ∞ to 0
secant	increases from to $-\infty$ to -1	secant	increases from 1 to ∞
cosecant	increases from 1 to ∞	cosecant	decreases from ∞ to 1

	III *quadrant*		IV *quadrant*
sine	decreases from 0 to -1	sine	increases from -1 to 0
cosine	increases from -1 to 0	cosine	increases from 0 to 1
tangent	increases from 0 to ∞	tangent	increases from $-\infty$ to 0
cotangent	decreases from ∞ to 0	cotangent	decreases from 0 to $-\infty$
secant	decreases from -1 to $-\infty$	secant	decreases from ∞ to 1
cosecant	increases from $-\infty$ to -1	cosecant	decreases from -1 to $-\infty$

Remark

In the above table we see the symbol ∞. Observe that ∞ is not a real number and is just a symbol. Statement like 'tan θ increases from 0 to ∞ for $\theta \in (0, \frac{\pi}{2})$ simply means that tan θ increases as θ increases in the interval $(0, \frac{\pi}{2})$ and assumes arbitraty large positive values as θ increase to $\frac{\pi}{2}$. Similarly, to say that cosecant decreases from -1 to $-\infty$ in the fourth quadrant means that cosec θ is a decreasing function for $\theta \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as θ approaches 2π.

13.4 Trigonometric Identities

An equation involving trigonometric functions which is true for all those angles for which the functions are defined is called a trigonometric identity. For example, $\sec\theta = \frac{1}{\cos\theta}$ is a trigonometric identity. It holds for all θ except those for which $\cos\theta = 0$.

An equation of the form $\sin\theta = \cos\theta$ is a trigonometric equation but not a trigonometric identity because it is not true for all θ. Trigonometric identities and solutions of trigonometric equations are very important and are useful in various problems of engineering and science.

Fundamental Identities

$$\sin\theta = \frac{1}{\operatorname{cosec}\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

All of the above identities are very easy to prove and the proofs are left as an exercise. From these we also observe that given one of the six trigonometric functions, all others can be found numerically and their signs can be found by seeing in which quadrant the angle lies.

Example 13.3

Given $\cot\theta = \frac{12}{5}$, θ in quadrant III, find the values of the other five functions.

Solution

$$\tan\theta = \frac{5}{12}, \ \sec^2\theta = 1 + \tan^2\theta = 1 + \frac{25}{144} = \frac{169}{144}$$

Now in quadrant III, sin θ, cos θ, sec θ and cosec θ are all negative. Therefore

$\sec\theta = -\frac{13}{12}$, $\cos\theta = -\frac{12}{13}$, $\sin\theta = \tan\theta\cos\theta = \frac{5}{12}\left(\frac{-12}{13}\right) = \frac{-5}{13}$,
$\operatorname{cosec}\theta = \left(\frac{-13}{5}\right)$

Example 13.4

Prove that $\sqrt{\frac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

Solution

Rationalise L.H.S. to obtain

$$\text{L.H.S.} = \frac{1-\sin A}{\sqrt{1-\sin^2 A}} = \frac{1-\sin A}{\cos A} = \sec A - \tan A$$
$$= \text{R.H.S.}$$

Example 13.5

Prove that

$$\frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta} = \sec\theta\operatorname{cosec}\theta + \cot\theta$$

Solution

$$\text{L.H.S.} = \frac{\sin\theta}{1-\cos\theta} + \frac{\tan\theta}{1+\cos\theta}$$
$$= \frac{\sin\theta + \sin\theta\cos\theta + \tan\theta - \tan\theta\cos\theta}{1-\cos^2\theta}$$
$$= \frac{\sin\theta + \sin\theta\cos\theta + \tan\theta - \sin\theta}{\sin^2\theta}$$
$$= \frac{\sin\theta\cos\theta + \tan\theta}{\sin^2\theta}$$
$$= \cot\theta + \sec\theta\operatorname{cosec}\theta = \text{R.H.S.}$$

Example 13.6

Prove that

$$\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1+\sin A}{\cos A}$$

Solution

$$\text{L.H.S.} = \frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{\sin A + 1 - \cos A}{\sin A - 1 + \cos A}$$

$$= \frac{(\sin A + 1 - \cos A)}{(\sin A + \cos A - 1)} \quad \frac{(\sin A + 1 + \cos A)}{(\sin A + 1 + \cos A)}$$

$$= \frac{(\sin A + 1)^2 - \cos^2 A}{(\sin A + \cos A)^2 - 1} = \frac{1 + 2\sin A + \sin^2 A - \cos^2 A}{\sin^2 A + \cos^2 A + 2\sin A\cos A - 1}$$

$$= \frac{2\sin A + 2\sin^2 A}{2\sin A\cos A}$$

$$= \frac{1 + \sin A}{\cos A} = \text{R.H.S.}$$

EXERCISE 13.2

Find the values of the other five functions in each of the following problems:

1. $\cos\theta = -\frac{1}{2}$, θ in quadrant II
2. $\sin\theta = \frac{3}{5}$, θ in quadrant I
3. $\tan\theta = \frac{3}{4}$, θ in quadrant III

Prove the following trigonometric identities:

4. $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$
5. $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta} = 2\,\text{cosec}\,\theta$
6. $\sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}} = \text{cosec}\,\theta - \cot\theta$
7. $\dfrac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \text{cosec}^2\theta$
8. $\dfrac{\text{cosec}\,\theta}{\cot\theta + \tan\theta} = \cos\theta$
9. $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$
10. $\dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$
11. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

12. $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)\,(1-2\sin^2\theta\cos^2\theta)$

13. $2\sec^2\theta - \sec^4\theta - 2\,\mathrm{cosec}^2\theta + \mathrm{cosec}^4\theta = \cot^4\theta - \tan^4\theta$

13.5 Cosine of the Difference of Two Angles

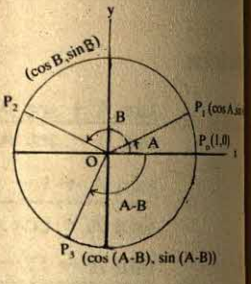


Fig. 13.10

We begin by establishing a formula for the cosine of the difference of two angles in terms of sines and cosines of the individual angles. We shall see that this helps us in proving several other important identities.

Recall that the terminal side of any angle cuts the circle with centre at O an unit radius at a point whose coordinates are respectively the cosine and sine of th angle. In the Fig. 13.10, OP_1 and OP_2 are the terminal sides of the angles A and respectively and OP_3 has been drawn to be the terminal side of the angle $A-B$. It i now clear that the chords P_0P_3 and P_1P_2 subtend the central angles of same size an hence are equal in length. Therefore, we obtain $[\cos(A-B)-1]^2 + \sin^2(A-B) =$ $(\cos B - \cos A)^2 + (\sin B - \sin A)^2$. This on simplification yields $\cos(A-B) = \cos A\cos B + \sin A\sin B$.

The method of proof of the above formula applies to all values of A and B Hence, the formula holds for all angles A and B, positive, zero or negative.

Example 13.7

Find the values of cos 15° and cos 75°.

Solution

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ\cos 30^\circ + \sin 45^\circ\sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Similarly, $\cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

Functions of 90° Minus Angle

$$\cos(90^\circ - \theta) = \cos 90^\circ\cos\theta + \sin 90^\circ\sin\theta$$
$$= \sin\theta$$

$$\sin(90^\circ - \theta) = \cos(90^\circ - (90^\circ - \theta)) = \cos\theta$$

Hence, we have

$\sin(90^\circ - \theta) = \cos\theta$ $\qquad$ $\mathrm{cosec}(90^\circ - \theta) = \sec\theta$

$\cos(90^\circ - \theta) = \sin\theta$ $\qquad$ $\sec(90^\circ - \theta) = \mathrm{cosec}\,\theta$

$\tan(90^\circ - \theta) = \cot\theta$ $\qquad$ $\cot(90^\circ - \theta) = \tan\theta$

From the above, replacing θ by $-\theta$ and recalling the formulas on section no. 13.3, we obtain

$\sin(90^\circ + \theta) = \cos\theta$ $\qquad$ $\mathrm{cosec}(90^\circ + \theta) = \sec\theta$

$\cos(90^\circ + \theta) = -\sin\theta$

$\tan(90^\circ + \theta) = -\cot\theta$

$\sec(90^\circ + \theta) = -\operatorname{cosec}\theta$

$\cot(90^\circ + \theta) = -\tan\theta$

Formulas for Functions of Sum and Difference of Two Angles

$$\begin{aligned}
\cos(A+B) &= \cos(A-(-B)) \\
&= \cos A\cos(-B) + \sin A\sin(-B) \\
&= \cos A\cos B - \sin A\sin B \\
\sin(A-B) &= \cos(90^\circ - (A-B)) \\
&= \cos((90^\circ - A) + B) \\
&= \cos(90^\circ - A)\cos B - \sin(90^\circ - A)\sin B \\
&= \sin A\cos B - \cos A\sin B \\
\sin(A+B) &= \sin(A-(-B)) \\
&= \sin A\cos(-B) - \cos A\sin(-B) \\
&= \sin A\cos B + \cos A\sin B
\end{aligned}$$

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} \\
&= \frac{\tan A + \tan B}{1 - \tan A\tan B} \qquad \text{(divide numerator and denominator by } \cos A\cos B)
\end{aligned}$$

Similar computations yield

$$\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}$$

Using above formulas, we get

$\sin(180^\circ - \theta) = \sin\theta$

$\cos(180^\circ - \theta) = -\cos\theta$

$\tan(180^\circ - \theta) = -\tan\theta$

$\sin(180^\circ + \theta) = -\sin\theta$

$\cos(180^\circ + \theta) = -\cos\theta$

$\tan(180^\circ + \theta) = \tan\theta$

$\operatorname{cosec}(180^\circ - \theta) = \operatorname{cosec}\theta$

$\sec(180^\circ - \theta) = -\sec\theta$

$\cot(180^\circ - \theta) = -\cot\theta$

$\operatorname{cosec}(180^\circ + \theta) = -\operatorname{cosec}\theta$

$\sec(180^\circ + \theta) = -\sec\theta$

$\cot(180^\circ + \theta) = \cot\theta$

From this we conclude that the period of the tangent function is π. All these can be directly deduced from the definitions given in section 13.3.

Also,

$2\sin A\cos B=\sin(A+B)+\sin(A-B)$
$2\cos A\sin B=\sin(A+B)-\sin(A-B)$
$2\cos A\cos B=\cos(A+B)+\cos(A-B)$
$2\sin A\sin B=-\cos(A+B)-\cos(A-B)$
These are called product formulas.

We also have the following sum formulas.

$$\sin A+\sin B=\sin\left(\frac{A+B}{2}+\frac{A-B}{2}\right)+\sin\left(\frac{A+B}{2}-\frac{A-B}{2}\right)$$
$$=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

Similarly

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Values of functions at 2A, $\frac{1}{2}$ A and 3A

$$\sin 2A=\sin(A+A)=2\sin A\cos A$$
$$\begin{aligned}\cos 2A=\cos(A+A)&=\cos^2 A-\sin^2 A\\&=2\cos^2 A-1\\&=1-2\sin^2 A\end{aligned}$$
$$\begin{aligned}\tan 2A&=\tan(A+A)\\&=\frac{2\tan A}{1-\tan^2 A}\end{aligned}$$

From the above $2\cos^2 A=1+\cos 2A$ and
$2\sin^2 A=1-\cos 2A$

Hence, replacing A by $\frac{A}{2}$, in the above formula, we get

$$\sin\frac{1}{2}A=\pm\sqrt{\frac{1-\cos A}{2}}$$
$$\cos\frac{1}{2}A=\pm\sqrt{\frac{1+\cos A}{2}}$$
$$\tan\frac{1}{2}A=\pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

Also $\tan\frac{1}{2}A=\frac{1-\cos A}{\sin A}=\frac{\sin A}{1+\cos A}$

Again $\sin 3A=\sin(A+2A)$

$$\begin{aligned}&= \sin A \cos 2A + \cos A \sin 2A\\ &= \sin A\,(1-2\sin^2 A) + 2 \sin A\,(1-\sin^2 A)\\ &= 3\sin A - 4\sin^3 A\end{aligned}$$

Similarly we can show that

$\cos 3A = 4\cos^3 A - 3\cos A$

$\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$ · The student is advised to prove these.

Values of sine and cosine for Some Special Angles

We have already found the values of cos 15° and cos 75°. Recall that once we know the values of cos 15° and cos 75°, it is easy to find the values of sin 15° and sin 75°. Thus

$\sin 15° = \sin(90° - 75°) = \cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

and $\sin 75° = \cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

Trigonometric Functions of an Angle of 18°

Let $\theta = 18°$. Then $2\theta = 90° - 3\theta$.
Therefore

$$\sin 2\theta = \cos 3\theta$$

or $$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta\,.$$

Since $\cos\theta \neq 0$, we get

$$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta$$

or $$4\sin^2\theta + 2\sin\theta - 1 = 0$$

Hence $\sin\theta = \dfrac{-2 \pm \sqrt{4+16}}{2\times 4} = \dfrac{-1\pm\sqrt{5}}{4}$

Since $\theta = 18°$, $\sin\theta > 0$. Therefore $\sin 18° = \dfrac{\sqrt{5}-1}{4}$

Also $\cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1 - \dfrac{6-2\sqrt{5}}{16}} = \sqrt{\dfrac{10+2\sqrt{5}}{16}}$

Hence, $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$.

Now we can easily find cos 36°, sin 36°.

$$\begin{aligned}\cos 36° = 1 - 2\sin^2 18° &= 1 - \frac{2(6-2\sqrt{5})}{16}\\ &= \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}.\end{aligned}$$

Hence, $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$

$$\text{Also } \sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \sqrt{1-\frac{6+2\sqrt{5}}{16}}$$
$$= \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

Example 13.8

Prove that
$\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$

Solution

$\cos 15^\circ = \sin(90^\circ - 15^\circ) = \sin 75^\circ$
$\cos 105^\circ = \cos(90^\circ + 15^\circ) = -\sin 15^\circ$
Therefore $\cos 105^\circ + \cos 15^\circ = \sin 75^\circ - \sin 15^\circ$.

Example 13.9

Prove that

$$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan x}{\tan x + \tan y}$$

Solution

$$\text{L.H.S.} = \frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y}$$

Dividing the numerator and denominator by $\cos x \cos y$, we get that

$$\text{L.H.S.} = \frac{\tan x - \tan y}{\tan x + \tan y} = \text{R.H.S}$$

Example 13.10

Prove that

$$\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A$$

Solution

$$\text{L.H.S.} = \frac{2\cos 4A \sin A}{2\cos 4A \cos A} = \tan A = \text{R.H.S.}$$

Example 13.11

Prove that

$$\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{2}\left[2\cos 2\theta \cos\frac{\theta}{2} - 2\cos 3\theta \cos\frac{9\theta}{2}\right]\\
&= \frac{1}{2}\left[\cos\left(2\theta + \frac{\theta}{2}\right) + \cos\left(2\theta - \frac{\theta}{2}\right) - \cos\left(3\theta + \frac{9\theta}{2}\right) - \cos\left(3\theta - \frac{9\theta}{2}\right)\right]\\
&= \frac{1}{2}\left[\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} - \cos\frac{15\theta}{2} - \cos\frac{3\theta}{2}\right]\\
&= \frac{1}{2}\left[\cos\frac{5\theta}{2} - \cos\frac{15\theta}{2}\right]\\
&= \sin\frac{5\theta+15\theta}{4}\ \sin\frac{15\theta-5\theta}{4}\\
&= \sin 5\theta \sin\frac{5\theta}{2} = \text{R.H.S.}
\end{aligned}$$

Example 13.12

Prove that

$$\frac{\sin 2A}{1+\cos 2A} = \tan A$$

Solution :

$$\text{L.H.S.} = \frac{2\sin A\cos A}{2\cos^2 A} = \tan A = \text{R.H.S.}$$

Example 13.13

Prove that

$$\sin 4A = 4\sin A\ \cos^3 A - 4\cos A\sin^3 A$$

Solution

$$\begin{aligned}
\text{L.H.S.} &= 2\sin 2A\cos 2A\\
&= 4\sin A\cos A\,(\cos^2 A - \sin^2 A)\\
&= 4\sin A\cos^3 A - 4\cos A\sin^3 A\\
&= \text{R.H.S.}
\end{aligned}$$

Example 13.14

Find the value of tan 22° 30′

Solution

Let $\theta = 45°$. Then $\frac{\theta}{2} = 22° \, 30'$

Now $\tan\frac{\theta}{2} = \frac{\sin\theta/2}{\cos\theta/2} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$

$$= \frac{\sin\theta}{1+\cos\theta} = \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}+1}$$

$$= \frac{1}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = \sqrt{2}-1.$$

Hence, $\tan 22° \, 30' = \sqrt{2}-1$.

EXERCISE 13.3

1. If $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, find the values of $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$ and $\tan(\alpha+\beta)$
2. Prove that
   $\cos(45° - A)\cos(45° - B) - \sin(45° - A)\sin(45° - B) = \sin(A+B)$
3. Show that
   $\sin 105° + \cos 105° = \cos 45°$
4. Prove that
   $\sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A = \cos A$.

Prove that

5. $\frac{\tan(45°+x)}{\tan(45°-x)} = \left[\frac{1+\tan x}{1-\tan x}\right]^2$
6. $\frac{\sin A+\sin 3A}{\cos A+\cos 3A} = \tan 2A$
7. $\frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta} = 4\cos 2\theta\cos 4\theta$
8. $\frac{\sin A+\sin B}{\cos A+\cos B} = \tan\left(\frac{A+B}{2}\right)$
9. $\cos 4x = 1 - 8\sin^2 x\cos^2 x$
10. $\tan 4\theta = \frac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}$

11. $(\sin 3A - \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$

12. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

Find sine, cosine and tangent of $\frac{x}{2}$ if $0 \leq x \leq 2\pi$ in the following problems:

13. $\tan x = -\frac{4}{3}$, x in Q II

14. $\cos x = -\frac{1}{3}$, x in Q III

15. $\sin x = \frac{1}{4}$, x in Q II

16. Find $\sin 7\frac{1}{2}^\circ$, $\cos 7\frac{1}{2}^\circ$ and $\tan 11\frac{1}{4}^\circ$

Prove that

17. $\sin^2 72^\circ - \sin^2 60^\circ = \frac{\sqrt{5}-1}{8}$

18. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

Prove that

19. $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10} = -\frac{1}{2}$

20. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{\alpha-\beta}{2}$

21. $\cos^2 A + \cos^2(A + 120^\circ) + \cos^2(A - 120^\circ) = \frac{3}{2}$

22. $\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x} = \cot 3x$

23. $\sin 3A + \sin 2A - \sin A = 4\sin A\cos\frac{A}{2}\cos\frac{3A}{2}$

24. $\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$

25. $\tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A$.

13.6 Tables of Trigonometric Functions

In order to deal with many problems in trigonometry, it is necessary to find the values of trigonometric functions for various angles. Trigonometric functions of any angle can be computed to any desired degree of accuracy. Tables are available which give approximate values of the sine and tangent of angles from 0° to 90°. The values of $\cos\theta$ and $\cot\theta$ can also be easily read out by using formulas like $\sin(90^\circ - \theta) = \cos\theta$, $\tan(90^\circ - \theta) = \cot\theta$; $\sec\theta$ and $\operatorname{cosec}\theta$ can be found out by noticing that

$\sec\theta = \dfrac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$. There are tables which give the values of six trigonometric functions of angles from 0° to 90°. For angles larger than 90° we use various formulas to reduce the value of a trigonometric function to the numerical value of some trigonometic function of an angle between 0° and 90°. For example $\sin 124° = \sin(90°+34°) = \cos 34°$. If sine of some angle is not given in the table we can use linear interpolation to find its value. We illustrate these by some examples.

Example 13.15

Find cot 131° 20′.

Solution

First we observe
$\cot(131°\,20') = -\tan 41°\,20'$
We are using $\cot(90°+\theta) = -\tan\theta$
By table, we see that $\tan 41°\,20' = .8796$.
Hence $\cot(131°\,20') = .8796$

Example 13.16

Find the angle θ if $\sin\theta = .7071$.

Solution

In the table of sines, we see that $\sin 45° = .7071$.
Hence $\theta = 45°$.

Example 13.17

Find the value of sin 23° 26′.

Solution

From the table we see that
$\sin 23°\,20' = .3961$
and $\sin 23°30' = .3987$ $\therefore$ difference $= .0026$
difference on 10′ is .0026.

Hence, the difference for $6' = \dfrac{6}{10} \times .0026 = .00156$

$= .0016$ (approx.)

Hence, $\sin 23^\circ\ 26' = .3961 + .0016$
$= .3977.$

Example 13.18

Find θ if $\cot\theta = .5750$.

Solution

$\tan(90^\circ - \theta) = \cot\theta = .5750.$
From table we see $.5735 = \tan 29^\circ 50'$
and $.5774 = \tan 30^\circ$
difference $= .0039$ Also $.5750 - .5735 = .0015$
For .0039 difference angle is $10'$

For .0015 difference angle is $\dfrac{10\times15}{39} = 4'$ approx.

Hence $90^\circ - \theta = 29^\circ\ 54'$. Therefore $\theta = 60^\circ\ 6'$.

EXERCISE 13.4

1. Find the following:
   - (i) $\cos 20^\circ\ 10'$
   - (ii) $\sin 48^\circ$
   - (iii) $\tan 54^\circ\ 30'$
   - (iv) $\cot 33^\circ\ 40'$
2. Find the angle $\theta, 0^\circ \le \theta \le 90^\circ$, if
   - (i) $\sin\theta = 0.5373$
   - (ii) $\cos\theta = .0087$
   - (iii) $\tan\theta = 34.37$
   - (iv) $\cot\theta = 3.018$
3. Find the following:
   - (i) $\sin 34^\circ\ 22'$
   - (ii) $\cos 64^\circ\ 34'$
   - (iii) $\tan 42^\circ\ 6'$
   - (iv) $\cot 47^\circ\ 26'$
4. Find θ where
   - (i) $\sin\theta = .5240$
   - (ii) $\cos\theta = .0424$
   - (iii) $\cot\theta = 1.246$
   - (iv) $\tan\theta = ..1362$

13.7 Graphs of Trigonometric Functions

We have already seen that all trigonometric functions are periodic. For example, the period of the sine and cosine functions is 2π, while that of the tangent function is π. Often, we have to deal with functions of the form $\sin ax$, $c\cos(ax+b)$ and so on. These functions are also periodic with period $\frac{2\pi}{a}$ as can be easily verified. For example the period of $5\sin(3x+4)$ is $\frac{2\pi}{3}$.

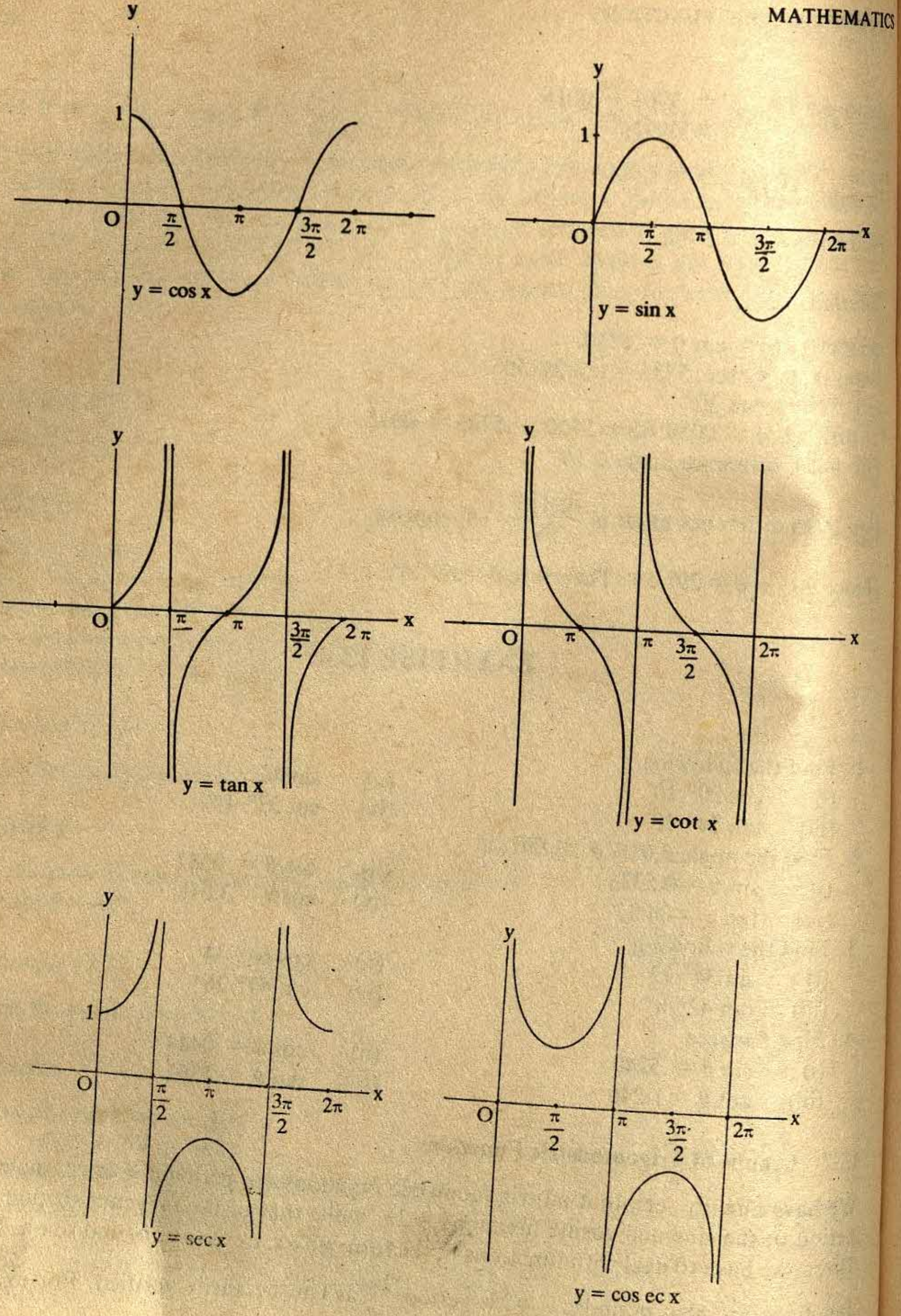

y
1
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
y = cos x
y
1
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
x
y = sin x
y
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
x
y = tan x
y
O
π
π
$\frac{3\pi}{2}$
2π
x
y = cot x
y
1
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
x
y = sec x
y
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
x
y = cos ec x

Fig.13.11

The graph of any periodic function with period T need to be sketched only in an interval of length T, as once it is drawn in one such interval, it can be easily drawn completely by repeating it over other intervals of length T.

The graph of $y = \sin x$

We have the following table

x	0°	30°	45°	60°	90°
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = .707$	$\frac{\sqrt{3}}{2} = .866$	1

We also have $\sin(\pi - t) = \sin t$ and $\sin(\pi + t) = -\sin t$. To draw the graph of $\sin x$ in $[0, 2\pi]$, we first draw it in $[0, \frac{\pi}{2}]$ using the above table, recalling that in this interval the function is increasing. Then in $[\frac{\pi}{2}, \pi]$, we draw it using $\sin(\pi - t) = \sin t$. Finally, in $[\frac{\pi}{2}, \pi]$, we draw it using the fact $\sin(\pi + t) = -\sin t$. The graph is now sketched in the figure (13.11). The graphs of some other trigonometric functions are also given.

Note: Observe that $\sin\theta = 0$ for $\theta = n\pi$ and $\cos\theta = 0$ for $\theta = (2n + 1)\frac{\pi}{2}$, where n is any integer.

Example 13.19

Graph of $3\cos 2x = f(x)$

Let $y = 3\cos 2x$

$-3 \leq y \leq 3$

Period of $f = \frac{2\pi}{2} = \pi$.

The graph of the curve in the interval $[0, \pi]$ is sketched in figure (13.12)

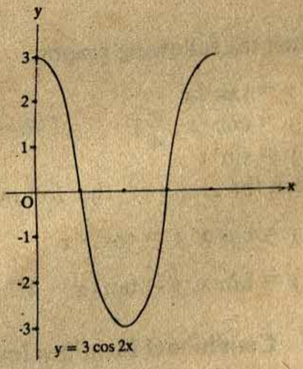


Fig. 13. 12

Example 13.20

$y = 3\sin(2x - 1) = f(x)$

Period of $f = \frac{2\pi}{2} = \pi$

Hence, range $-3 \leq y \leq 3$.

Suppose we wish to write

$y = 3\sin(2x - 1) = 3\sin 2t$

or $\quad 2x - 1 = 2t$

or $\quad t = x - \frac{1}{2}$

or $\quad x = t + \frac{1}{2}$

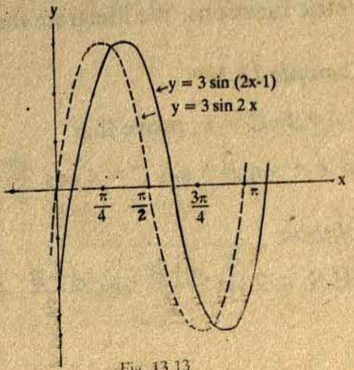


Fig. 13.13

Thus if we draw the graph of 3 sin 2t and 'shift' it by $\frac{1}{2}$ we get the required graph. The required graph is drawn in the figure (13.13).

Example 13.21

Sketch the graph of $y = \sin x$ and $y = \sin 2x$ on the same axes.

Solution

$y = \sin x$
Range $= \{y : -1 \leq y \leq 1\}$
Period $= 2\pi$
$y = \sin 2x$
Range $= \{y : -1 \leq y \leq 1\}$
Period $= \pi$
The graphs are drawn below.

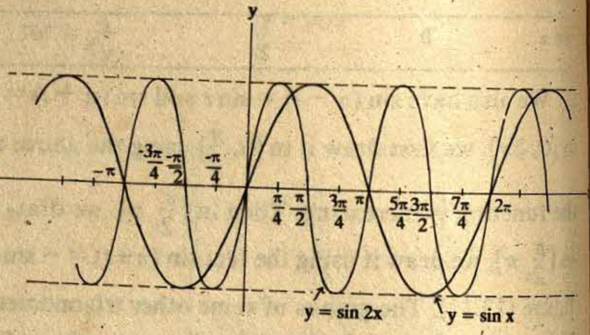


Fig. 13. 14

EXERCISE 13.5

Sketch the following graphs:

1. $y = \tan 3x$
2. $y = 3 \sin 2x$
3. $y = \cos (x - \frac{\pi}{4})$
4. $y = 3 \sin (3x + 1)$
5. $y = \sin^2 x$
6. $y = \cos^2 x$

Sketch the graph of the following pair of equations on the same axes:

7. $y = \cos x$, $y = \cos \frac{1}{2} x$
8. $y = \sin x$, $y = \sin (x + 45°)$
9. $y = \tan x$, $y = \tan (x - 45°)$
10. $y = \cos 2x$, $y = \cos (2x - \pi)$

13.8 Conditional Identities Involving the Angles of a Triangle

When *A, B, C* are angles of a triangle, many identities hold between their trigonometric functions. We illustrate the method of proof by some examples.

Example 13.22

If $A + B + C = \pi$, prove that

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

Solution

$$\text{L.H.S.} = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \sin C$$

$$= 2\sin\left(90^\circ - \frac{C}{2}\right)\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\left(90^\circ - \frac{C}{2}\right)\right)$$

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right)$$

$$= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)$$

$$= 4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cos\frac{C}{2} = \text{R.H.S.}$$

Example 13.23

If $A + B + C = \pi$, show that
$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

Solution

$$\begin{aligned}
\text{L.H.S.} &= 2\cos(A+B)\cos(A-B) + \cos 2C \\
&= 2\cos(\pi - C)\cos(A-B) + \cos 2C \\
&= -2\cos C\cos(A-B) + 2\cos^2 C - 1 \\
&= -1 + 2\cos C\{\cos C - \cos(A-B)\} \\
&= -1 + 2\cos C[\cos\{\pi - (A+B)\} - \cos(A-B)] \\
&= -1 - 2\cos C\{\cos(A+B) + \cos(A-B)\} \\
&= -1 - 2\cos C\{2\cos A\cos B\} \\
&= -1 - 4\cos A\cos B\cos C = \text{R.H.S.}
\end{aligned}$$

Example 13.24

If $A + B + C = \pi$, show that

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

Solution

$$\tan\frac{A+B}{2} = \tan\left(90^\circ - \frac{C}{2}\right) = \cot\frac{C}{2}$$

Also $\tan\dfrac{A+B}{2} = \dfrac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}}$

Hence, $\dfrac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \dfrac{1}{\tan\frac{C}{2}}$

or $\quad \tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$.

or $\quad \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$,
which is what we wanted to prove.

EXERCISE 13.6

If $A+B+C=\pi$, show that

1. $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$
2. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$
3. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
4. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$
5. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
6. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$
7. $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$
8. $\tan A+\tan B+\tan C=\tan A\tan B\tan C$
9. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
10. $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$

13.9 Trigonometric Equations

In solving trigonometric equations, we come across situations of the type $\sin\theta=\sin\alpha$ or $\cos\theta=\cos\alpha$ or $\tan\theta=\tan\alpha$. Before illustrating the technique of solving trigonometric equations by some examples, we find the general value of all angles having a given sine, cosine or tangent.

1. Suppose $\sin\theta=\sin\alpha$
   Then $\sin\theta-\sin\alpha=0$

   or $2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}=0$

   Hence, either $\cos\frac{\theta+\alpha}{2}=0$ or $\sin\frac{\theta+\alpha}{2}=0$.

   Therefore, either

   $\frac{\theta+\alpha}{2}$ = any odd multiple of $\pi/2$

   or $\frac{\theta-\alpha}{2}$ = any multiple of π (See section 13.7)

Thus either

$\theta = -\alpha +$ **any odd multiple of** π

or $\theta = \alpha +$ **any even multiple of** π

Thus the general value of θ **such that** $\sin\theta = \sin\alpha$ **is given by**

$\theta = n\pi + (-1)^n\alpha$, **where** n **is an integer.**

2. Suppose $\cos\theta = \cos\alpha$
   Then $\cos\theta - \cos\alpha = 0$

   Hence, $-2\sin\frac{\theta+\alpha}{2}\ \sin\frac{\theta-\alpha}{2} = 0.$

   Therefore,

   either $\frac{\theta+\alpha}{2}$ = any multiple of π

   or $\frac{\theta-\alpha}{2}$ = any multiple of π.

   Hence, the general solution for θ such that $\cos\theta = \cos\alpha$ is given by
   $\theta = 2n\pi \pm \alpha$

3. Suppose $\tan\theta = \tan\alpha$

   Then $\frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

   or $\sin\theta\cos\alpha - \sin\alpha\cos\theta = 0$
   or $\sin(\theta - \alpha) = 0$
   or $\theta - \alpha = n\pi$.

   Hence, the general solution for θ satisfying $\tan\theta = \tan\alpha$ is
   $\theta = n\pi + \alpha$

Example 13.25

Solve the equation
$\sin\theta + \sin 3\theta + \sin 5\theta = 0$

Solution

Given $\sin\theta + \sin 3\theta + \sin 5\theta = 0$
or $(\sin\theta + \sin 5\theta) + \sin 3\theta = 0$

or $2\sin\frac{5\theta+\theta}{2}\cos\frac{5\theta-\theta}{2} + \sin 3\theta = 0$

or $2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$
or $\sin 3\theta(2\cos 2\theta + 1) = 0$

or $\sin 3\theta = 0$ or $\cos 2\theta = -\frac{1}{2}$

When $\sin 3\theta = 0$ then $3\theta = n\pi$

or $\theta = \frac{n\pi}{3}$. When $\cos 2\theta = -\frac{1}{2}$, then $2\theta = 2n\pi \pm \frac{2\pi}{3}$

or $\theta = n\pi \pm \frac{\pi}{3}$. This yields $\theta = (3n+1)\frac{\pi}{3}$ or $\theta = (3n-1)\frac{\pi}{3}$

These solutions are contained in the solution set of $\sin 3\theta = 0$.
Hence the required solution set is given by $\{\theta : \theta = \frac{n\pi}{3}, n \text{ an integer}\}$.

Example 13.26

Solve $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$

Solution

Divide the equation by 2 to get

$$\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}}$$

or $\cos\frac{\pi}{6}\cos\theta + \sin\frac{\pi}{6}\sin\theta = \cos\frac{\pi}{4}$

or $\cos(\frac{\pi}{6}-\theta) = \cos\frac{\pi}{4}$ or $\cos(\theta-\frac{\pi}{6}) = \cos\frac{\pi}{4}$

Hence the solution is $\theta - \frac{\pi}{6} = 2m\pi \pm \frac{\pi}{4}$

or $\theta = 2m\pi \pm \frac{\pi}{4} + \frac{\pi}{6} = 2m\pi + \frac{\pi}{4} + \frac{\pi}{6}$

or $2m\pi - \frac{\pi}{4} + \frac{\pi}{6}$

or $\theta = 2m\pi + \frac{5\pi}{12}$

or

$2m\pi - \frac{\pi}{12}$.

Note: The above method can be used to solve equations of the form $a\cos\theta + b\sin\theta = c$

Divide the equation by $\sqrt{a^2+b^2}$ to get

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta = \frac{c}{\sqrt{a^2+b^2}}$$

Let $\tan\alpha = \frac{b}{a}$. Then

$\sin \alpha = \frac{b}{\sqrt{a^2+b^2}}, \cos \alpha = \frac{a}{\sqrt{a^2+b^2}}.$

The equation now becomes

$\cos \alpha \cos \theta + \sin \alpha \sin \theta = \frac{c}{\sqrt{a^2+b^2}}$

Hence, $\cos (\theta - \alpha) = \frac{c}{\sqrt{a^2+b^2}}$

This will have solutions if

$\left| \frac{c}{\sqrt{a^2+b^2}} \right| \leq 1$. Because, in this case, we can find $(\theta - \alpha) = \beta$ (say), so that

$\cos \beta = \frac{c}{\sqrt{a^2+b^2}}.$

The equation can be easily solved now.

Example 13.27

Solve $2 \cos^2 t + 3 \sin t = 0$.

Solution

The equation yields

$2(1 - \sin^2 t) + 3 \sin t = 0$
or $2 \sin^2 t - 3 \sin t - 2 = 0$
or $(2 \sin t + 1) (\sin t - 2) = 0$.
Hence, either $2 \sin t + 1 = 0$
or $\sin t - 2 = 0$. But this last situation cannot occur. Hence, $\sin t = -\frac{1}{2} = \sin \frac{7\pi}{6}$.
Therefore, the solution is $\theta = n\pi + (-1)^n \frac{7\pi}{6}$.

EXERCISE 13.7

Solve the following equations:

1. $\cos \theta + \cos 3\theta - 2 \cos 2\theta = 0$
2. $\sin 2\theta + \cos \theta = 0$
3. $\sec^2 2\theta = 1 - \tan 2\theta$.
4. $\sin \theta = \tan \theta$
5. $\sin 3\theta + \cos 2\theta = 0$
6. $\sin x + \cos x = \sqrt{2} \cos A$
7. $4 \cos \theta - 3 \sec \theta = \tan \theta$
8. $\sin m\theta + \sin n\theta = 0$
9. $2 \tan \theta - \cot \theta = -1$
10. $\cot^2 x + \frac{3}{\sin x} + 3 = 0$.

CHAPTER 14

SOLUTION OF TRIANGLES

The three sides and the three angles of a triangle are called its parts. The process of finding the unknown parts of a triangle from the given parts of a triangle is called solving the triangle. Triangle solving has many application in surveying, navigation, astronomy and other sciences. A particularly interesting application is that of determining heights and distances which are not easy to measure.

14.1 Some Basic Formulas

Let the angles of a triangle be denoted by A, B and C and the sides opposite to them by a, b and c respectively.

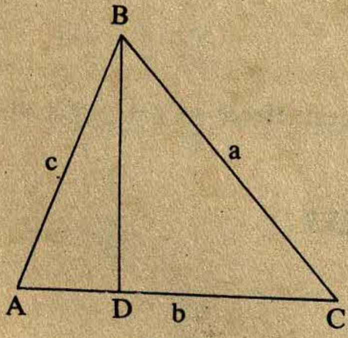


Fig.14.1

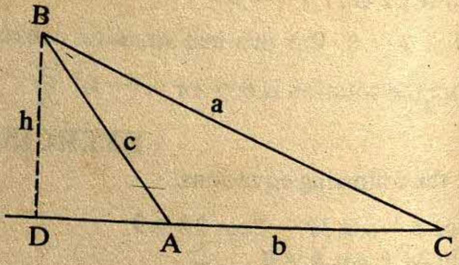


Fig.14.2

From Figs. 14.1 and 14.2, we see that

$$\frac{h}{c} = \sin A \text{ and } \frac{h}{a} = \sin C.$$

These two relations hold whether the angle A is acute or obtuse. It obviously holds if A is a right angle. Thus $c \sin A = a \sin C$ or $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$.

Similarly, drawing the altitude from the vertex *C*, we can show that

$$\frac{\sin A}{a} = \frac{\sin B}{b}.$$

Therefore, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

These equations constitute the *Law of Sines.* From this we see that in any triangle, the length of the sides are proportional to the sines of the opposite angles.

The Law of Cosines

Referring back to Fig. 14.1, we have

$$\begin{aligned} AB^2 &= BD^2 + AD^2 \\ &= BD^2 + (AC)^2 - CD)^2 \\ &= BD^2 + AC^2 + CD^2 - 2AC\,.\,CD \end{aligned}$$

Now, $BD^2 + CD^2 = BC^2$ and $CD = BC \cos C$.

Hence,

$$AB^2 = AC^2 + BC^2 - 2AC\,.\,BC \cos C$$

or $$c^2 = a + b^2 - 2ab \cos C \qquad \text{...(14.1)}$$

The same equation is obtained if the angle *A* is obtuse as in figure 14.2. We also have two other equations obtained in a similar way:

$$b^2 = a^2 + c^2 - 2ac \cos B \qquad \text{...(14.2)}$$

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \text{...(14.3)}$$

The equations (14.1), (14.2), and (14.3) costitute the law of cosines. A convenient form of the cosine law, when the angles are to be found, is

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Half Angles

We have

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

Hence,

$$\begin{aligned} 2\sin^2\frac{A}{2} &= 1 - \cos A \\ &= \frac{a^2-(b-c)^2}{2bc} \\ &= \frac{(a+b-c)\,(a-b+c)}{2bc} \end{aligned}$$

Let $a + b + c = 2s$.

Then $\sin^2 \dfrac{A}{2} = \dfrac{4(s-b)(s-c)}{4bc}$

or $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$ ... (14.4)

Similarly, we obtain

$$\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}} \qquad \text{...(14.5)}$$

and $\sin \dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$ ... (14.6)

Also,

$$2\cos^2 \frac{A}{2} = 1 + \cos A$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}$$

Hence, as before

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \qquad \text{...(14.7)}$$

$$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} \qquad \text{...(14.8)}$$

$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} \qquad \text{(14.9)}$$

Using formulas (14.4) to (14.9), we have

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad \text{...(14.10)}$$

$$\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \qquad \text{...(14.11)}$$

$$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \qquad \text{...(14.12)}$$

The Area of a Triangle

Referring again to Figs 14.1 and 14.2, we see that the area Δ of the triangle ABC is given by

$$\Delta = \frac{1}{2} bc \sin A.$$

Similarly, we can show that $\Delta = \dfrac{1}{2} ca \sin B = \dfrac{1}{2} ab \sin C$.

Now,

$$\Delta = \frac{1}{2} bc \sin A = bc \sin \frac{A}{2} \cos \frac{A}{2}$$

$$= bc\sqrt{\frac{(s-b)\,(s-c)}{bc}}\ \sqrt{\frac{s(s-a)}{bc}} \quad \text{or } \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

This formula is known as Hero's formula.

Example 14.1

Given $A = 25$, $b = 52$, $c = 63$.
Find $\cos A$.

Solution

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{52^2+63^2-25^2}{2\times52\times63} = \frac{2704+3969-625}{2\times52\times63}$$

$$= \frac{12}{13}.$$

Hence, $\cos A = \frac{12}{13}$.

Example 14.2

Given $a = 15$, $b = 36$, $c = 39$.
Find $\sin \frac{A}{2}$.

Solution

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}};\ 2s = 90$$

$$= \sqrt{\frac{(45-36)\,(45-39)}{26\times 39}}$$

$$= \sqrt{\frac{9\times 6}{36\times 39}} = \sqrt{\frac{1}{26}}$$

EXERCISE 14.1

Given $a = 18$, $b = 24$, $c = 30$.
Find (i) $\sin A$, $\sin B$, $\sin C$
(ii) $\tan A$, $\tan B$, $\tan C$
(iii) Δ
(iv) $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$

14.2 Some More Formulas

1. In any triangle ABC, we have
$a = b \cos C + c \cos B$
$b = c \cos A + a \cos C$
$c = a \cos B + b \cos A.$

Proof: Referring again to fig. 14.1, we have
$b = AD + DC$, $AD = c \cos A$ and $DC = a \cos C$.
Hence $b = c \cos A + a \cos C$.
If A is obtuse(See fig. 14.2),we have

$$b = CD - AD,\ CD = a \cos C,\ AD = c \cos (\pi - A) = -c \cos A$$

Hence $b = a \cos C - (-c \cos A) = a \cos C + c \cos A$.
Other formulas are proved in similar manner.

2. *Law of tangents*: In a triangle ABC,

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

Proof: By law of sines, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k, \text{ say. Therefore,}$$

$$\frac{b-c}{b+c} = \frac{k(\sin B - \sin C)}{k(\sin B + \sin C)} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}$$

$$= \tan \frac{B-C}{2} \frac{\cos\left(\frac{\pi}{2} - \frac{A}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)} = \tan \frac{B-C}{2} \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}$$

or $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$.

We have similar formulas for $\tan \frac{C-A}{2}$ and $\tan \frac{A-B}{2}$. The above formula can also be written as

$$\frac{b-c}{b+c} = \frac{\tan \frac{B-C}{2}}{\tan \frac{B+C}{2}}$$

The student is advised to prove it.

EXERCISE 14.2

1. Write the formulas for $\tan \frac{C-A}{2}$ and $\tan \frac{A-B}{2}$ analogous to that of $\tan \frac{B-C}{2}$ and prove them.

2. Prove that

$$\frac{a+b}{c} = \frac{\cos \frac{A-B}{2}}{\sin \frac{C}{2}}, \qquad \frac{a-b}{c} = \frac{\sin \frac{A-B}{2}}{\cos \frac{C}{2}}$$

Half Angle Formulas

ABC is a triangle. Let AO, BO and CO be angle bisectors. Then $OD = OE = OF = r$, where r is the radius of the incircle. From Fig. 14.3 we see that $\tan \frac{A}{2} = \frac{r}{AF}$, $\tan \frac{B}{2} = \frac{r}{BD}$ and $\tan \frac{C}{2} = \frac{r}{CE}$.

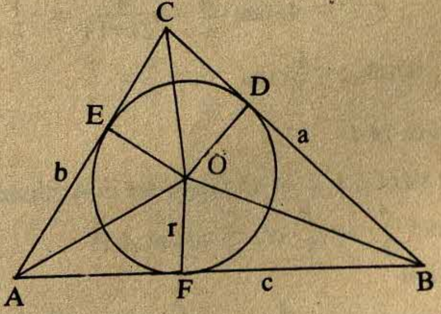


Fig.14.3

We shall find expressions for r, AF, BD and CE in terms of the sides a, b and c. The area Δ of the triangle ABC is the sum of the areas of the triangles AOB, BOC and AOC. Hence,

$$\Delta = \frac{1}{2}cr + \frac{1}{2}ar + \frac{1}{2}br = \frac{1}{2}(a+b+c)r = rs.$$

Therefore, $r = \frac{\Delta}{s}$.

Also triangle AFO and AEO are congruent. Hence, $AF = AE$. Similarly, $BF=BD$ and $CE = CD$. Since the sum of these six segments is $a + b + c = 2s$, we get

$AF + BD + CD = s$.

Therefore $AF = s - (BD + CD) = s-a$.

Similarly, $BD = s-b$ and $CE = s-c$.

Hence,

$$\tan \frac{A}{2} = \frac{r}{s-a}, \tan \frac{B}{2} = \frac{r}{s-b} \text{ and } \tan \frac{C}{2} = \frac{r}{s-c}.$$

These formulas give a quick way of finding the angles of a triangle, given its sides.

Example 14.3

If $a = 13, b = 14, c = 15$, find $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$, r and Δ.

Solution

$$s = \frac{13+14+15}{2} = 21$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \times 8 \times 7 \times 6}$$

$$= \sqrt{7 \times 7 \times 6 \times 6 \times 4} = 84 \text{ sq. unit}$$

$$r = \frac{\Delta}{s} = 4. \tan \frac{A}{2} = \frac{4}{21-13} = \frac{1}{2}$$

$$\tan B/2 = \frac{4}{7}, \tan \frac{C}{2} = \frac{4}{6} = \frac{2}{3}.$$

Example 14.4

If the sides a, b, c of a triangle are in Arithmetic Progression, prove that

$\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in $A.P.$

Solution

To show that $\cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2}$.

$$\text{L.H.S.} = \cot \frac{A}{2} + \cot \frac{C}{2}$$

$$= \frac{s-a}{r} + \frac{s-c}{r} = \frac{2s-(a+c)}{r} = \frac{2s-2b}{r} = 2\left(\frac{s-b}{r}\right) = 2 \cot \frac{B}{2}.$$

Example 14.5

Let R be the radius of the circumcircle of a triangle ABC. Show that

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}\,.$$

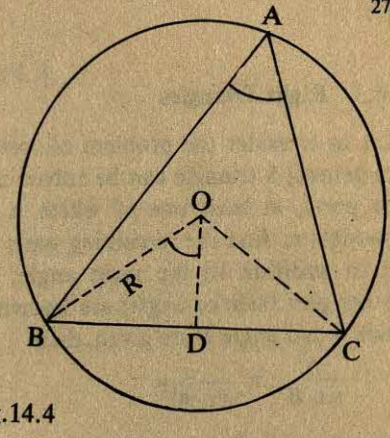


Fig.14.4

Solution

Let O be the circumcentre and OD be the perpendicular on BC from O. Then $BD = CD = \frac{a}{2}\cdot$ Also triangles OBD and OCD are congruent. Furthermore, $\angle BOC = 2A$. Hence, $\angle BOD = A$. Therefore, $\sin A = \frac{BD}{BO}=\frac{\frac{a}{2}}{R}$ Hence, $R = \frac{a}{2\sin A}$.

Hence, by law of sines $R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}$

EXERCISE 14.3

In any triangle ABC, prove that

1. $\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$
2. $a(b\cos C - c\cos B) = b^2 - c^2$
3. $a(\cos C - \cos B) = 2(b - c)\cos^2\frac{A}{2}$
4. $\frac{\sin(B-C)}{\sin(B+C)}=\frac{b^2-c^2}{a^2}$
5. $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$
6. Find the angles A, B, C and also the radius of incircle, given
   (i) $a = 18$ $b = 24$ $c = 30$
   (ii) $a = 13$ $b = 4$ $c = 15$
   Use trigonometric tables, if necessary.
7. Let R be the radius of the circumcircle of a triangle ABC. Show that $R=\frac{abc}{4\Delta}$.
8. The sides of a triangle are 22, 28, 36 centimetres. Find the areas of the inscribed circle, the triangle and the circumscribed circle.

14.3 Right Triangles

Let us consider the problem of solving right triangles. In general a triangle can be solved if we know three of its parts, at least one of which is a side. Thus it is possible to find the remaining parts of a right triangle if in addition to the right angle, one side and any other part (side or angle) are known. Thus if $A = 90°$, side b and angle B are given, then

$$\frac{b}{\sin B} = \frac{a}{\sin 90°}$$

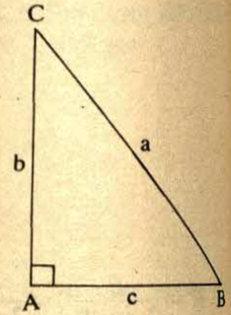


Fig.14.5

Hence $a = \frac{b}{\sin B}$. Also $\frac{c}{b} = \cot B$, so that $c = b \cot B$. If the sides b and c are given then $a = \sqrt{b^2 + c^2}$. Similarly, we can solve the triangle in the other cases.

Example 14.6

A right triangle has
$c = 64, \angle A = 61°$ and $\angle C = 90°$. Find the remaining parts.

Solution

Since $\angle A + \angle B + \angle C = 180°$, we have
$\angle B = 180° - (61° + 90°) = 29°$.

Now $\frac{b}{\sin B} = \frac{c}{\sin 90°}$.

Hence $b = c \sin B = 64 \sin 29°$
or $b = 64 \times .4848 = 31.0272$ (from tables).
Also $a = 64 \cos 29° = 64 \times .8746 = 55.9744$

Example 14.7

Solve the right triangle ABC, given $a = 45$, $b = 35.2$ and $\angle C = 90°$.

Solution

$\tan B = \frac{b}{a} = \frac{35.2}{45}$.

From tables, $B = 38° 1'$ approximately.
Therefore $A = 51° 59'$
Also

$$\frac{c}{a} = \sec B \text{ or } c = 45 \times 1.296$$
$$= 58.32.$$

EXERCISE 14.4.

Solve the right triangles, where $\angle C = 90°$.

1. $a = 43, A = 53°$
2. $c = 6.5, A = 26°$
3. $a = 50.4, b = 26.2$
4. $b = 3.3, c = 4.4$
5. $b = 4.5, A = 39°$
6. $a = 412, c = 610$

14.4 Oblique Triangles

We now consider the problem of solving oblique triangles - the triangles which are not right angled. As already stated, we can solve a triangle if we know three of its parts at least one of which is a side. Different cases to be considered are:

Case I: The three sides are given.
Case II: Two sides and the included angles are given.
Case III: Two sides and the angle opposite to one of them are given.
Case IV: One side and two angles are given

Case I: Given the three sides a, b and c

Since a, b, c are given, $s = \frac{a+b+c}{2}$ is known. Then the half angles can be computed by using formulas for their sines, cosines or tangents. Only two angles need be found, the third can be found by subtracting the sum of the two from 180°. The angles can also be found by using cosine formulas. These are usually avoided unless a, b and c are small numbers, as it may involve longer calculations.

Example 14.8

Solve the triangle, given
$a = 20, b = 30$ and $c = 21$

Solution

$s = \frac{20+30+21}{2} = 35.5$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{35.5\,(35.5-20)}{30\times21}}$$

$\log\cos\frac{A}{2} = \frac{1}{2}\,[\log 35.5 + \log 15.5 - \log 30 - \log 21]$

$$= \frac{1}{2}\,[\,1.5502 + 1.19035 - 1.4771 - 1.3222]$$

$$= \frac{1}{2}\,[\,2.7405 - \;2.7993]$$

$$= \frac{1}{2}\,[-\;.0588] = -.0294$$

Logarithmic cosine is defined by

$L \cos \frac{A}{2} = 10 + \log \cos \frac{A}{2}$ (Other logarithmic trigonometric functions are defined similarly and there are tables which give their vàlues).

Hence L $\cos \frac{A}{2} = 10 - .0294$

$= 9.9706$

From tables $\frac{A}{2} = 20^\circ\ 50'$

Hence $A = 41^\circ\ 40'$

$$\text{Now} \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} = \sqrt{\frac{35.5 \times 5.5}{20 \times 21}}$$

$$= \sqrt{\frac{195.25}{420}}$$

$$\log \cos \frac{B}{2} = \frac{1}{2}[\,2.2905 - 2.6233\,] = -\frac{1}{2}[.3328]$$

$$= -.1664$$

$L \cos \frac{B}{2} = 9.8336$. Hence, $\frac{B}{2} = 47^\circ$ so that $B = 94^\circ$.

Therefore, $C = 180^\circ - (A+B) = 44^\circ\ 20'$

EXERCISE 14. 5

1. If $a = 9$, $b = 10$, $c = 11$, find B, given $\log 2 = .30103$, L $\tan 29^\circ\ 29' = 9.7523472$ and L $\tan 29^\circ\ 30' = 9.7526420$
   Hint: Use formula for $\tan \frac{B}{2}$ and interpolate.
2. Solve the triangle, given
   $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} - 1$
   [Hint: use cosine formaula.]
3. The sides of a triangle are 2, 3 and 4.
   Find the greatest angle, if
   $\log 2 = .30103$, $\log 3 = .4771213$
   L $\tan 52^\circ\ 14' = 10.1108395$
   and L $\tan 52^\circ\ 15' = 10.1111004$
4. Making use of the tables, solve the triangle, given
   $a = 25$, $b = 26$ and $c = 27$.

Case II: Given two sides b and c and the included angle A

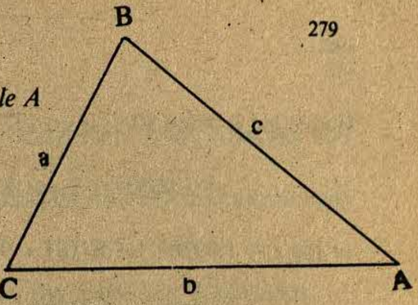


Fig.14.6

Let b denote the greater side.
Then we have

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

This gives us the angle $\frac{B-C}{2}$.

Since $\frac{B+C}{2}=90°-\frac{A}{2}$,

we now get the angles B and C. The third side can now be determined by Sine law. The third side can also be found by Cosine law, but as mentioned earlier, we avoid it unless the sides are small in numbers.

Example: 14.9

Solve the triangle, given

$$b=50,\ c=80,\ A=132°$$

Solution

$$\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}$$
$$=\frac{30}{130}\cot 66°$$

Hence

$$\mathrm{L}\tan\frac{C-B}{2}=\log 3+\mathrm{L}\cot 66°-\log 13$$
$$=.47712+9.64858-1.11394$$
$$=10.12570-1.11394=9.01176$$

Hence $\frac{C-B}{2}=5°\ 50'$ or $C-B=11°\ 40'$.

Also $\frac{C+B}{2}=90°-66°=24°$ or $C+B=48°$.

Therefore, $C=29°\ 50'$, $B=18°\ 10'$

Now $\frac{a}{\sin A}=\frac{c}{\sin C}$. Hence

$$a=\frac{c\sin A}{\sin C}=\frac{80\sin 132°}{\sin 29°\ 50'}$$

Hence, $\log a = \log 80 + \text{L} \sin 132^\circ - \text{L} \sin 29^\circ\ 50'$

$$= 1.90309 + \text{L} \cos 42^\circ - \text{L} \sin 29^\circ\ 50'$$

or $\log a = 1.90309 + 9.87107 - 9.69677$

$$= 11.77416 - 9.69677$$

$$= 2.07739$$

Hence, $a = 119.6$ (from tables).

EXERCISE 14.6

1. If $a = 21$, $b = 11$ and $C = 34^\circ\ 42'\ 30''$, find A and B, given
   $\log 2 = .30103$,
   $\text{L} \tan 72^\circ\ 38'\ 45'' = 10.50515$.
2. If $b = 14$, $c = 11$ and $A = 60^\circ$, find B and C, given
   $\log 2 = .30103$, $\log 3 = .4771213$
   $\text{L} \tan 11^\circ\ 44' = 9.3174299$
   $\text{L} \tan 11^\circ\ 45' = 9.3180640$
3. If $a = \sqrt{3} + 1$, $b = \sqrt{3} - 1$ and $C = 60^\circ$, find the other side and angles.
4. If $a = 40$, $c = 40\sqrt{3}$ and $B = 30^\circ$, solve the triangle.

Case III: Given two sides b and c and the angle B opposite to one of them

In this case there may be no triangle, or one triangle or two triangles depending on the given parts. For this reason, it is called the ***ambiguous case***. These may be considered as follows:

By cosine law
$b^2 = c^2 + a^2 - 2ca \cos B$.
or $a^2 - 2ac \cos B = b^2 - c^2$. Adding $c^2 \cos^2 B$ to both sides, we get
$a^2 - 2ac \cos B + c^2 \cos^2 B$
$= b^2 - c^2 + c^2 \cos^2 B$
$= b^2 - c^2 \sin^2 B$.

Thus $(a - c \cos B)^2 = b^2 - c^2 \sin^2 B$.
or $a = c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B}$.
This equation helps us to determine a, given b, c and B.

(i) If $b < c \sin B$, then $b^2 - c^2 \sin^2 B < 0$ and hence there is no solution and hence no triangle.

(ii) If $b = c \sin B$, then $a = c \cos B$ (only one solution) and the triangle is right angled. Note that $\frac{b}{\sin B} = c \cdot$ Hence $\angle C = 90°$

(iii) If $b > c \sin B$, then there are two solutions of (1). But, since $a > 0$, the value $c \cos B - \sqrt{b^2 - c^2 \sin^2 B}$ is inadmissible unless it is positive, i. e. unless

$\sqrt{b^2 - c^2 \sin^2 B} < c \cos B$
or $b^2 - c^2 \sin^2 B < c^2 \cos^2 B$
or $b^2 < c^2$.

Hence there are two triangles only when $b > c \sin B$ and only one triangle if $b < c$.

(iv) If B is obtuse, then $c \cos B$ is negative, and one value of a is negative and the corresponding triangle is impossible. The other value will be positive only when

$c \cos B + \sqrt{b^2 - c^2 \sin^2 B} > 0$

or $\quad \sqrt{b^2 - c^2 \sin^2 B} > -c \cos B$

or $\quad b^2 - c^2 \sin^2 B > c^2 \cos^2 B.$

or $\quad b^2 > c^2$ or $b > c.$

Hence, if B is obtuse there is no triangle if $b < c$ and only one triangle if $b > c$.

Example 14.10

Given $a = 2528$, $b = 3126$, $B = 51° \, 25'$, solve the triangle.

Solution

In the discussion, we had used the letters b, c and B for the given parts. When other letters are used (as in this case) the reasoning remains the same. In this case the side b (opposite to angle B) $> a$ and form (iii), we have exactly one triangle.

Now, $\frac{a}{\sin A} = \frac{b}{\sin B}$

Hence, $\sin A = \frac{a}{b} \sin B.$

Therefore,

$\text{L} \sin A = \log 2528 + \text{L} \sin 51° \, 25' - \log 3126$

$= 3.40278 + 9.89304 - 3.49499$ from tables)

$= 13.29582 - 3.49499$

$$= 9.80083$$

Hence, $A = 39^\circ\ 13'$. Therefore, $C = 180^\circ - (51^\circ\ 25' + 39^\circ\ 13')$
$$= 89^\circ\ 22'$$

Also $c = \dfrac{b \sin C}{\sin B}$.

Hence, $\log c = \log 3126 + \text{L} \sin 89^\circ\ 22' - \text{L} \sin 51^\circ\ 25'$

$$= 3.49499 + 9.99997 - 9.89304$$

$$= 3.60192$$

Hence, $c = 3999$.

Example 14.11

Solve the triangle, $b = 50$, $c = 63$ and $B = 54^\circ$.

Solution

Let us calculate $c \sin B$ to see whether $b \lesssim c \sin B$ or not. In fact we calculate $\dfrac{c \sin B}{b} = \sin C$.

Now
$$\log \sin C = \log 63 + \text{L} \sin B - 10 - \log 50$$
$$= 1.79934 + 9.90796 - 10 - 1.69897$$
$$= .00833$$

Since $\log \sin C > 0$, we conclude $\sin C > 1$. This implies $c \sin B > b$ and there is no solution.

Example 14.12

Solve the triangle $a = 28$, $b = 38$ and $A = 39^\circ$.

Solution

Here $a < b$ and $b \sin A = 38 \times .62932 < a$.
By (iii) there are two solutions.

Now, $c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}$

$\log (b \cos A) = \log 38 + \text{L} \cos A - 10 = 1.57978 + 9.89050 - 10$
$= 11.47028 - 10 = 1.47028$
Hence, $b \cos A = 29.99$

Similarly,

$\log (b \sin A) = 1.57978 + 9.70887 - 10$

$= 11.37865 - 10 = 1.37865$

and $b \sin A = 23.91$. Also $2 \log (b \sin A) = 2.75730$.

Therefore, $b^2 \sin^2 A = 571.9$

Hence, $a^2 - b^2 \sin^2 A \doteq 784 - 571.9 = 212.1$

Hence, $\sqrt{a^2 - b^2 \sin^2 A} = 14.59$.

Hence, $c = 29.99 \pm (14.59)$

$= 44.59$ or 15.40

Now,

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

Hence, $\sin B = \frac{38}{28} \sin 39°$.

Hence,

$L \sin B = 1.57978 + 9.79887 - 1.44716$

$= 11.37865 - 1.44716$

$= 9.93149$

Hence, $B = 58° 40'$

and $C = 180° - (A + B)$

$= 82° 20'$

So the solutions are

$a = 28 \quad b = 38 \quad c = 44.58$

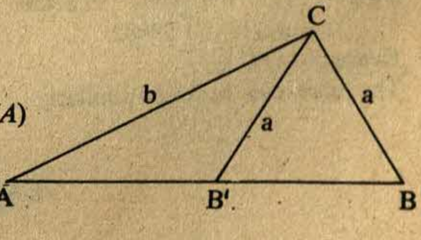
$A = 29° \quad B = 58° 40' \quad C = 82° 20'$

For the other solution, $c' = 15.40$

$B' = 180° - B = 121° 20'$ and $C = 180° - (B' + A)$

$= 180° - (121° 20' + 39°)$

$= 19° 40'$.



The figure is of the form as shown.

Fig.14.7

EXERCISE 14.7

1. If $a = 5$, $b = 7$ and $\sin A = \frac{3}{4}$, solve the triangle, if possible.
2. If $A = 30°$, $a = 100$, $c = 100\sqrt{2}$. solve the triangle.
3. If $A = 30°$, $b = 8$ and $a = 6$, find c.
4. If a, b and A are given so that two triangles may be formed, show that the angles B and B' in the two triangles are supplementary to each other.

5. If $a = 5$, $b = 4$ and $A = 45°$, find the other angles, having given

$\log 2 = .30103$, $L\sin 34° 26' = 9.7523919$
and $L\sin 34° 27' = 9.7525761$.

Case IV: Given one side and two angles, say, a B and C

If two angles of a triangle are given then third can be found easily. Thus
$A = 180° - (B+C)$.
The other sides can be found out by using sine formula. Thus

$$b = a\,\frac{\sin B}{\sin A} \text{ and } c = a\,\frac{\sin C}{\sin A}.$$

Example: 14.13

Solve the triangle given
$c = 1.732$, $A = 23° 16'$, $B = 20° 3'$.

Solution

$$C = 180° - (23° 16' + 20° 3')$$
$$= 136° 41'$$

$$a = \frac{c}{\sin c} \sin A.$$

Hence,
$\log a = .23855 + 9.59661 - 9.83634$
$= -.00118 = \bar{1}.99882.$
Hence, $a = .9973$.
The side b can be found similarly.

EXERCISE 14.8

1. Solve the triangles given
   (i) $c = 72$, $A = 56°$, $B = 65°$
   (ii) $a = 18$, $B = 108°$, $A = 25°$.
   (iii) $b = 302$, A, $= 50° 10'$, $C = 72°$.
2. The base angle of a triangle are $22\frac{1°}{2}$ and $112\frac{1°}{2}$. Show that the base is equal to twice the height.
3. The angles of a triangle are 1:2:7. Show that the ratio of the greatest side to the least side is $\sqrt{5} + 1 : \sqrt{5} - 1$.

14.5 Heights and Distances

In this section we shall discuss some problems regarding heights and distances. We begin by defining some terms which are needed in our discussion.

Angles of Elevation and Angles of Depression

Suppose O and P are two points; P being at a higher level than O Let OM be the horizontal line drawn through O to meet at M, the vertical line drawn through P. The angle MOP is called the *angle of elevation* of the point P as seen from O Draw PN, parallel to OM, so that PN is the horizontal line passing through P. The angle NPO is called the *angle of depression* of the point O as seen from P.

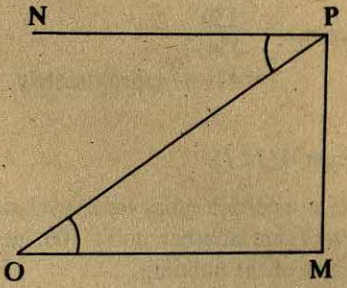


Fig.14.8

Bearings of a point

Let NS stand for a line in the north south direction and EW in the east-west direction. The acute angle which OA makes with NS is called the bearing of the point A from O. The bearing of A may be precisely indicated by giving the size of the angle and specifying whether it is measured from ON or OS and whether to the east or west. Thus in the figure OA is in the direction 30° east of north and the bearing is written $N\ 30°\ E$. The bearings of some other points are also indicated in the figure.

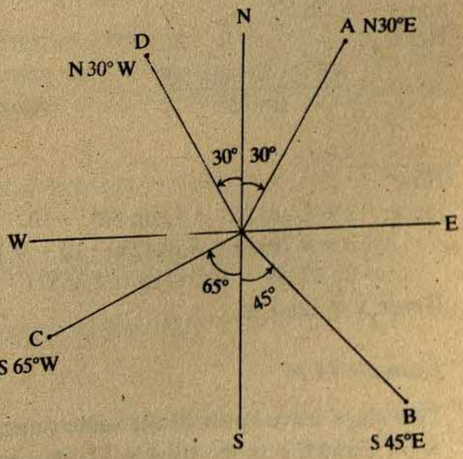


Fig.14.9

Example on Heights and Distances

Example 14.14

From a cliff 150 metre above the shore line the angle of depression of a ship is 19° 30′.Find the distance from the ship to a point on the shore directly below the observer.

Solution

We want to find x.

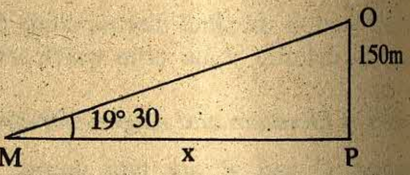


Fig.14.10

Now, $\frac{150}{x} = \tan 19° \ 30'$

$= .35412$

Hence, $x = \frac{150}{.35412}$

$= 421$ m approximately.

Example 14.15

From a certain point on a level street, the angle of elevation of the top of a building is 50°From another point 100 metres away, the angle of elevation is 38°. Find the height of the building.

Solution

We want to find x.

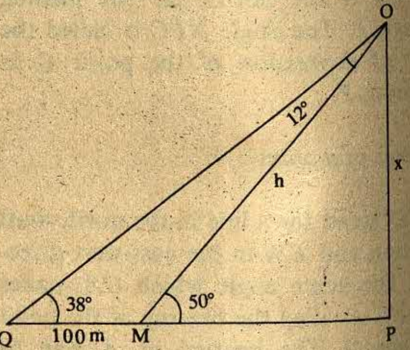


Fig.14.11

$$\frac{h}{\sin 38°} = \frac{100}{\sin 12°}$$

and $x = h \sin 50°$

Hence,

$$x = \frac{100 \sin 38° \sin 50°}{\sin 12°}$$

Now

$\log x = 2 + \mathrm{L} \sin 38° + \mathrm{L} \sin 50° - 10 - \mathrm{L} \sin 12°$

$= 2 + 9.78934 + 9.88425 - 10 - 9.31788$

$= 21.67359 - 19.31788 = 2.35571.$

Hence, $x = 226.8$ m

Example 14.16

Two ships leave a port at the same time. One goes 24 km per hour in the direction N 38°Eand the other travels 32 km. per hour in the direction S 52° E. Find the distance between the ships at the end of 3 hours.

Solution

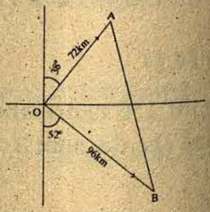


Fig.14.12

Note that $\angle AOB = 90°$

Hence, $AB = \sqrt{72^2 + 96^2}$

$= 24 \times 5 = 120$ km

Example 14.17

The horizontal distance between two towers is 60 m and the angular depression of the top of the first as seen from the top of the second, which is 150 m high is 30° Find the height of the first.

Solution

$BD = 150$ m

To find h.

Now $\frac{150-h}{60} = \tan 30° = \frac{1}{\sqrt{3}}$

or $h\sqrt{3} = 150\sqrt{3} - 60$

$h = 150 - 20\sqrt{3}$

$= 115.359$ m.

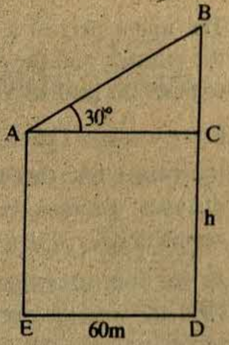


Fig.14.13

EXERCISE 14.9

1. The angle of elevation of a ladder leaning against a house is 58°, and the foot of the ladder is 9.6 m from the house. Find the length of the ladder.
2. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°; when he retreats 20 m from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river.
3. From a tower 128 m high, the angle of depression of a car is 26° 10′. Find how far the car is from the tower.
4. At a point A, the angle of elevation of a tower is such that its tangent is $\frac{5}{12}$; on walking 240 m nearer the tower, the tangent of the angle of elevation is $\frac{3}{4}$. Find the height of the tower.
5. To find the distance between two points A and B on opposite sides of a river, a surveyor runs along a line AC perpendicular to AB. By measurement, he finds $AC = 200$ m and $\angle ACB = 48° \; 40'$. Find the distance AB.
6. Find the height of a chimney when it is found that, on walking towards it 50 m in a horizontal line through its base, the angular elevation of its top changes from 30° to 45°.
7. A town B is 13 km South and 18 km West of a town A. Find the bearing and distance of B from A.

8. An observer on the top of a cliff.200 m above the sea-level, observes the angles of depressions of two ships on opposite sides of the cliff to be 45° and 30° respectively. Find the distance between the ships if the line joining them points to the base of the cliff.

9. The upper part of a tree broken over by the wind makes an angle of 30° with the ground, and the distance from the root to the point where the top of the tree touches the ground is 10 m.What was the height of the tree?

10. From a tower 126 m high, the angles of depression of two rocks which are in a horizontal line through the base of the tower are 16° and 12° 20′. Find the distance between the rocks if they are on (a) the same side of the tower; (b) opposite sides of the tower.

11. At the foot of a mountain the elevation of its summit is 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain.

CHAPTER 15

INVERSE TRIGONOMETRIC FUNCTIONS

15.1 The Inverse of a Function

Let $f : X \rightarrow Y$ be a function. Recall that f is said to be *one-to-one* if $f(x_1) \neq f(x_2)$, whenever $x_1 \neq x_2$. We say that f is *onto* if for each $y \in Y$, there exists an $x \in X$ such that $y = f(x)$. If $f : X \rightarrow Y$ is one-to-one and onto then, we can define a unique function $g: Y \rightarrow X$ such that $g(y) = x$, where $x \in X$ is such that $y = f(x)$. Thus the domain of g = range of f and range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}. Note that the graphs of f and f^{-1} are

$\{(x, f(x)) \mid x \in \text{domain } f\}$ and
$\{(f(x), x) \mid x \in \text{domain } f\}$ respectively.

Example 15.1

Let $f(x) = 2x, 0 \leq x \leq 1$.

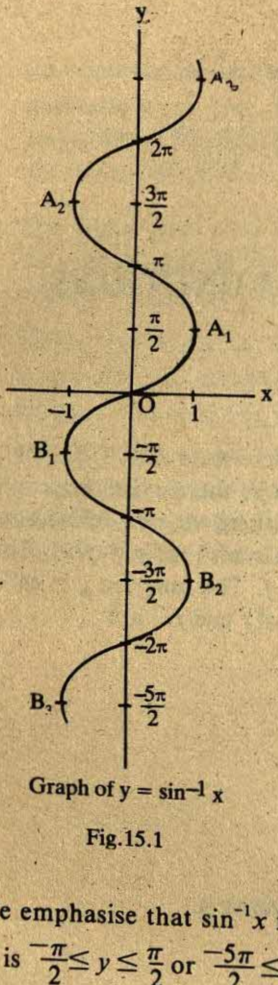
Then $f^{-1}(x) = \frac{x}{2}, 0 \leq x \leq 2$. The student is advised to verify it.

Inverse of Trigonometric Functions

Let us consider the function $f(x) = \sin x$. We know that if $\sin x = \sin \theta$, then

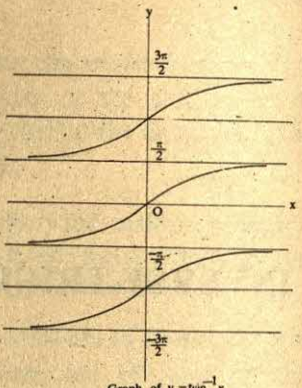
$$x = n\pi + (-1)^n \theta \text{ for some integer } n.$$

However, if we restrict the function $\sin x$ to $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$, then this function is one-to one and its image is $-1 \leq y \leq 1$. Actually, $\sin x$ restricted to any of the intervals $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}, \frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}, -\frac{5\pi}{2} \leq x \leq -\frac{3\pi}{2}$ etc, is one-to-one and its image is $-1 \leq y \leq 1$. We therefore conclude that in each of these intervals we can define the inverse of the sine function. Note that $y = \sin^{-1} x$ means $x = \sin y$. Furthermore, $\sin^{-1} x$ should not be confused with $(\sin x)^{-1}$ which is equal to $\frac{1}{\sin x}$. Also, note that $\sin^{-1} x$ is an angle whose sine is x.



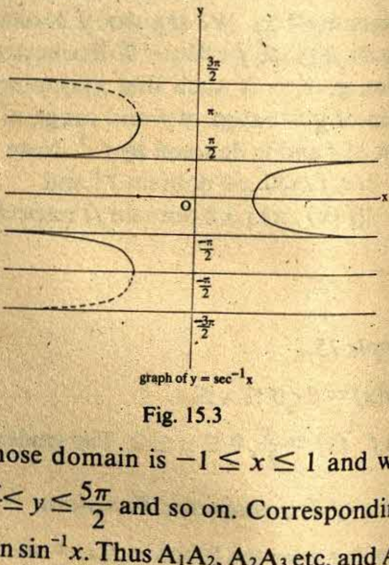
Graph of $y = \sin^{-1} x$

Fig.15.1



Graph of $y = \tan^{-1} x$

Fig. 15.2



graph of $y = \sec^{-1} x$

Fig. 15.3

We emphasise that $\sin^{-1}x$ is a function whose domain is $-1 \le x \le 1$ and whose range is $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ or $\frac{-5\pi}{2} \le y \le \frac{-3\pi}{2}$ or $\frac{3\pi}{2} \le y \le \frac{5\pi}{2}$ and so on. Corresponding to each such interval, we get a branch of the function $\sin^{-1}x$. Thus A_1A_2, A_2A_3 etc. and A_1B_1, B_1B_2 etc. are different branches of $\sin^{-1}x$. Of course these are graphs of different functions.

From the graph, we see that for each x, $-1 \le x \le 1$, there exists a unique $y \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ such that $\sin y = x$. This y is called the principal value of $\sin^{-1}x$. Notice that $\sin^{-1} : [-1, 1] \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ becomes a function. We also note that the principal value of $\sin^{-1}x$ is the least numerical value among all values of $\sin^{-1}x$. The considerations similar to the above are valid for other inverse trigonometric functions like $\cos^{-1}x$, $\tan^{-1}x$ and so on.

Remark:

The principal value of $\sin^{-1}x$, for $x > 0$, is precisely the length of the arc of unit circle centred at origin which subtends an angle at the centre whose sine is x. For this reason $\sin^{-1}x$ is also denoted by arc sin x. Similarly, $\cos^{-1}x$, $\tan^{-1}x$ etc. are denoted by arc cos$x$, arc tan x and so on.

Following is a table giving the inverse trigonometric functions and their principal value branches:

Functions	*Principal Value Branches*
$y = \sin^{-1}x$	$\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ where $-1 \le x \le 1$
$y = \cos^{-1}x$	$0 \le y \le \pi, -1 \le x \le 1$
$y = \tan^{-1}x$	$\left\{\begin{array}{l} \frac{-\pi}{2} < y < \frac{\pi}{2}, -\infty < x < \infty \\ 0 \le y < \frac{\pi}{2}, 1 \le x < \infty \end{array}\right\}$
$y = \sec^{-1}x$	$\left\{\begin{array}{l} 0 < y \le \pi/2, 1 \le x < \infty \\ \frac{\pi}{2} < y \le \pi, -\infty < x \le -1 \end{array}\right\}$
$y = \text{cosec}^{-1}x$	$\left\{\begin{array}{l} 0 < y \le \frac{\pi}{2}, \quad 1 \le x < \infty \\ \frac{-\pi}{2} \le y < 0, -\infty < x \le -1. \end{array}\right\}$
$y = \cot^{-1}x$	$0 < y < \pi, -\infty < x < \infty$

Note:

1. When y is positive $(0 \le y \le 1)$, there are two angles, one between 0 and $\frac{\pi}{2}$ and the other between $\frac{-\pi}{2}$ and 0 having their cosines equal to y (Recall $\cos x$ is an even function). In this case we take smallest positive angle as the principal value of $\cos^{-1}y$.

   For example, $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$ and not $-30°$ even though $\cos(-30°) = \frac{\sqrt{3}}{2}$.
2. Whenever no branch of an inverse trigonometric function is specifically mentioned, we mean the principal branch of the function.
3. We have given the graph of $\sin^{-1}x$. The graphs of other inverse trigonometric functions can be drawn from the knowledge of the graphs of corresponding trigonometric functions. The graphs of $\tan^{-1}x$ and $\sec^{-1}x$ are given in Fig 15.2 and 15.3 respectively.

15.2 Properties of Inverse Trigonometric Functions

1. $\theta = \sin^{-1}(\sin\theta)$. Let $\sin\theta = x$
   Then $\theta = \sin^{-1}x = \sin^{-1}(\sin\theta)$
   Similarly, $\theta = \cos^{-1}(\cos\theta) = \tan^{-1}(\tan\theta)$
   $= \sec^{-1}(\sec\theta) = \text{cosec}^{-1}(\text{cosec}\,\theta)$
   $= \cot^{-1}(\cot\theta)$.

   Note that in the above formulas and the others to follow, assume that we are dealing with the principal branches of the functions. In no case, we should consider two different branches simultaneously while working with these formulas. For example, $\sin 135° = \frac{1}{2}$ and principal value of $\sin^{-1}\left(\frac{1}{2}\right) = 45°$. So we cannot conclude $45° = 135°$, since 135° lies in the branch corresponding to the interval $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, that is the branch which we should consider while considering $\sin^{-1}(\sin 135°)$ if we want to have $\theta = \sin^{-1}(\sin\theta)$.

2. $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$, $\cot^{-1}x = \tan^{-1}\frac{1}{x}$ and $\sec^{-1}x = \cos^{-1}\frac{1}{x}$. Let us prove, for example, $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$. Let $\text{cosec}^{-1}x = y$. Then $x = \text{cosec}\,y$ or $\frac{1}{x} = \sin y$ or $y = \sin^{-1}\left(\frac{1}{x}\right)$. Other formulas can be similarly proved.

3. Since $\sin^{-1}x$, $\tan^{-1}x$ and $\text{cosec}^{-1}x$ may assume positive or negative values, we determine the value of $\sin^{-1}(-x)$. Let $\sin^{-1}(-x) = y$ or $-x = \sin y$
   or $x = -\sin y = \sin(-y)$
   Hence, $-y = \sin^{-1}x$ or $y = -\sin^{-1}x$.
   Hence, $\sin^{-1}(-x) = -\sin^{-1}x$.
   We can similarly prove that $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $\tan^{-1}(-x) = -\tan^{-1}x$ and $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$.

4. We now prove the following important relations between the inverse trigonometric functions:

   (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

   (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

   (iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

   (iv) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, $xy < 1$

   (v) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

   (vi) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$.

Proof:

(i) Let $\sin^{-1}x = \theta$
Then $x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
or $\frac{\pi}{2} - \theta = \cos^{-1}x$.

Hence, $\sin^{-1}x + \cos^{-1}x = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}$
(ii) and (iii) are similarly proved.

(iv) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$.
Let $\tan^{-1}x = \theta_1$, $\tan^{-1}y = \theta_2$. Then $x = \tan\theta_1$ and $y = \tan\theta_2$

$$\text{Now, } \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 . \tan\theta_2}$$
$$= \frac{x+y}{1-xy}.$$

Hence, $\theta_1 + \theta_2 = \tan^{-1}\frac{x+y}{1-xy}$.

Note that if $xy \geq 1$, then (iv) cannot hold. If $xy = 1$, then R.H.S. of (iv) is not defined. Suppose $xy > 1$. If both $x, y < 0$, then $\frac{x+y}{1-xy} > 0$ and R.H.S. of (iv) is negative, while L.H.S. is positive. Therefore, (iv) cannot hold if $xy \geq 1$. However, if $xy < 1$, then we can show that $\theta_1 + \theta_2$ lies between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ (Problem 14, Exercise 15.1). Hence, in this case (iv) holds.

(v) We prove (v) in the same way as (iv). The only difference is that now we do not need any restriction, because if θ_1, θ_2 lie in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then so does their difference.

(vi) Let
$\tan^{-1}x = \theta$, then $x = \tan\theta$.

$$\text{Now, } \frac{2x}{1+x^2} = \frac{2\tan\theta}{1+\tan^2\theta}$$
$$= \frac{2\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta} = \sin 2\theta$$

Hence, $2\theta = \sin^{-1}\frac{2x}{1+x^2}$.

Similarly, we can prove the other relations. We must remark that these results hold only for restricted values of x. Thus $\sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1}x$ if $-1 \leq x \leq 1$. Notice that if $x > 1$ or $x < -1$, then $2\tan^{-1}x$ is either $> \frac{\pi}{2}$ or $< \frac{-\pi}{2}$. Similarly,

$\cos^{-1}\frac{1-x^2}{1+x^2}=2\tan^{-1}x$ if $0\leq x<\infty$ and $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$ if $-1<x<1$.

Example 15.1

Find the principal values of

(i) $\operatorname{cosec}^{-1}(-1)$

(ii) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$.

Solution

(i) **Let** $\operatorname{cosec}^{-1}(-1)=y$. **Then** y must satisfy $\frac{-\pi}{2}\leq y<0$ and $\operatorname{cosec} y=-1$.

This is true only for $y=\frac{-\pi}{2}$, which is, therefore, the principal value of $\operatorname{cosec}^{-1}(-1)$.

(ii) Let $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)=y$. Then $\cot y=-\frac{1}{\sqrt{3}}$ or $\tan y=-\sqrt{3}$. Now,

$\tan\theta=\tan(\pi-\theta)$. Therefore, since $\tan\frac{\pi}{3}=\sqrt{3}$, we must have principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ as $\pi-\frac{\pi}{3}=\frac{2\pi}{3}$.

Example 15.2

Find two branches, other than the principal value branch, of $\tan^{-1}x$.

Solution

By graph of $\tan x$, we can see that the inverse function of $\tan x$ exists in each of the intervals $\frac{-3\pi}{2}<y<\frac{-\pi}{2}$, $\frac{\pi}{2}<y<\frac{3\pi}{2}$, $\frac{3\pi}{2}<y<\frac{5\pi}{2}$ and so on. Hence, the two branches of $\tan^{-1}x$ can be taken to be

$\frac{-3\pi}{2}<\tan^{-1}x<\frac{-\pi}{2}$ and $\frac{\pi}{2}<y<\frac{3\pi}{2}$.

Example 15.3

Prove that

$2\tan^{-1}\left(\frac{1}{x}\right)=\sin^{-1}\frac{2x}{x^2+1}$.

Solution

Let $\tan^{-1}\frac{1}{x}=\theta$

or $\frac{1}{x} = \tan\theta$

Hence, L.H.S. $= 2\theta$

$$\text{R.H.S.} = \sin^{-1}\frac{2x}{x^2+1} = \sin^{-1}\frac{\frac{2}{x}}{1+\frac{1}{x^2}}$$

$$= \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = \text{L.H.S.}$$

Example 15.4

Prove that

$\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$.

Solution

Let $\tan^{-1}\frac{1}{7} = \theta_1$ and

$\tan^{-1}\frac{1}{13} = \theta_2$.

Then

$$\frac{1}{7} + \frac{1}{13} = \tan\theta_1 + \tan\theta_2$$

$$= \tan(\theta_1 + \theta_2)(1 - \tan\theta_1 . \tan\theta_2)$$

$$= \tan(\theta_1 + \theta_2)\left(1 - \frac{1}{7}\cdot\frac{1}{13}\right)$$

or $\frac{20}{91} = \frac{90}{91}\tan(\theta_1 + \theta_2)$

or $\tan(\theta_1 + \theta_2) = \frac{2}{9}$

or $\theta_1 + \theta_2 = \tan^{-1}\frac{2}{9}$.

Alternatively,

$$\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13}$$

$$= \tan^{-1}\frac{\frac{1}{7} + \frac{1}{13}}{1 - \frac{1}{7}\cdot\frac{1}{13}}$$

$$= \tan^{-1}\left(\frac{2}{9}\right).$$

Example 15.5

Solve $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$.

Solution

$\tan^{-1} 2x + \tan^{-1} 3x$

$$= \tan^{-1} \frac{2x+3x}{1-2x.3x}$$

$$= \tan^{-1} \frac{5x}{1-6x^2}, \text{ provided } 6x^2 < 1.$$

Hence $\tan^{-1} \frac{5x}{1-6x^2} = \frac{\pi}{4}$

or $\frac{5x}{1-6x^2} = 1$

or $6x^2 - 5x - 1 = 0$ or $(6x + 1)\quad(x - 1) = 0$

Hence, the required solution is $x = \frac{-1}{6}$.

Note that $x = 1$ does not satisfy the solution as $\tan^{-1}2 + \tan^{-1}3 > \frac{\pi}{2}$.

Example 15.6

Write $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ in the simplest form.

Solution

Let $\tan^{-1} \frac{\cos x}{1+\sin x} = \theta$.

Then $\frac{\cos x}{1+\sin x} = \tan\theta$

Now,

$$\frac{\cos x}{1+\sin x} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{(\cos \frac{x}{2} + \sin \frac{x}{2})^2} = \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}}$$

$$= \frac{\sin(\frac{\pi}{4} - \frac{x}{2})}{\cos(\frac{\pi}{4} - \frac{x}{2})} = \tan(\frac{\pi}{4} - \frac{x}{2}) = \tan\theta.$$

Hence, $\theta = \frac{\pi}{4} - \frac{x}{2}$

or $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}$.

Example 15.7

Show that

$$\tan^{-1} \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2.$$

Solution

Let $x^2 = \cos 2\theta$. Then

$$\text{L.H.S.} = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \theta\right)\right) = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x.$$

$$= \text{R.H.S.}$$

EXERCISE 15.1

1. Find the principal values of

   (i) $\sin^{-1}(-1)$ (ii) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ (iii) $\cos^{-1}\left(\frac{-1}{2}\right)$ (iv) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

   (v) $\cot^{-1}(\sqrt{3})$ (vi) $\text{cosec}^{-1}(-2)$

2. Find any three branches, other than the principal value branch, of each of the following:
   (i) $\sec^{-1} x$ (ii) $\text{cosec}^{-1} x$ (iii) $\cot^{-1} x$.

Prove that

3. $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$

4. $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

5. $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}$

6. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$

7. $\tan^{-1} t + \tan^{-1}\frac{2t}{1-t^2} = \tan^{-1}\frac{3t-t^3}{1-3t^2}, t > 0$

8. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5}$

9. $\cot^{-1}\frac{ab+1}{a-b} + \cot^{-1}\frac{bc+1}{b-c} + \cot^{-1}\frac{ca+1}{c-a} = 0$

10. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, prove that

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{x^2}{b^2} = \sin^2\alpha$$

11. Find the value of

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right).$$

12. Solve the following equations:

(i) $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

(ii) $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$.

13. Write the following in the simplest form:

(i) $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$

(ii) $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

(iii) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$

(iv) $\tan^{-1}\left(\frac{3a^2x-x^3}{a(a^2-3x^2)}\right)$

14. Let $\tan^{-1}x = \theta_1$ and $\tan^{-1}y = \theta_2$. Show that $\frac{-\pi}{2} < \theta_1 + \theta_2 < \frac{\pi}{2}$ provided $xy < 1$.

[Hint: If $\theta_1 > 0$ and $\theta_2 < 0$ or $\theta_1 < 0$ and $\theta_2 > 0$, then $\frac{-\pi}{2} < \theta_1 + \theta_2 < \frac{\pi}{2}$, since $\frac{-\pi}{2} < \theta_1 < \frac{\pi}{2}$ and $\frac{-\pi}{2} < \theta_2 < \frac{\pi}{2}$.

Suppose $\theta_1 > 0$, $\theta_2 > 0$. Then

$xy < 1 \Rightarrow x < \frac{1}{y}$ or $\tan\theta_1 < \frac{1}{\tan\theta_2}$

Hence, $\tan\theta_1 < \cot\theta_2 = \tan(\frac{\pi}{2} - \theta_2)$

Since, $\tan\theta$ is increasing in the interval $(0, \frac{\pi}{2})$, we get $\theta_1 < \frac{\pi}{2} - \theta_2$ or $\theta_1 + \theta_2 < \frac{\pi}{2}$.

Since $\theta_1 + \theta_2 > 0$, we have $\frac{-\pi}{2} < \theta_1 + \theta_2 < \frac{\pi}{2}$. If $\theta_1 < 0$ and $\theta_2 < 0$, then $\theta_1 + \theta_2 < 0$. Also $xy < 1 \Rightarrow \tan\theta_1 \tan\theta_2 < 1$

or $(-\tan\theta_1)(-\tan\theta_2) < 1$. Hence, $\tan(-\theta_1)\tan(-\theta_2) < 1$. But $-\theta_1 > 0$ and $-\theta_2 > 0$.

Hence, from above, we get $(-\theta_1) + (-\theta_2) < \frac{\pi}{2}$ or $\theta_1 + \theta_2 < \frac{-\pi}{2}$. This implies the result.]

CHAPTER 16

FREQUENCY TABLES

16.1 **Raw Data**

Statistics deals with information given in numerical terms; for example the amount of land owned by different peasants in a village, tehsil, or district, the monthly wage earned by workers in a factory; the numbers of male and female voters in a constituency; the numbers of people of different religions in a town; etc.

Such information is collected through *censuses* and *surveys*. The Government of India, for example, conducts an All India Census every ten years (the last was in 1981) to record the actual number of persons alive at a given time, along with their ages, sex, occupation, housing condition, etc. The *National Sample Survey* (or NSS) is a programme of periodically obtaining information, in which the Government is interested, to help it in framing its policies. For example, the NSS may be asked to collect information about the extent of unemployment, the requirement for housing, etc. The Ministry of Labour conducts *family budget surveys* to find out how much an average worker's family has to spend to obtain the minimum requirements of food, clothing, housing, education, medical aid, etc. This information is then used to calculate the *cost of living index* which measures the effect of changing prices on the standard of living.

The terms *census* and *survey* are used to distinguish between two different techniques of collecting information. In the All India Census information about age, sex, education, etc., is obtained about *every* person alive at that time. It is another matter that we may not succeed in covering the entire population, but every effort is made to do so. The Ministry of Labour does not attempt to contact *every* working class family to get the required information for calculating the cost of living index. Instead, it *selects* a group of workers to represent all the workers and then collects the information on family expenses from the workers in the selected group only.

The term *census* is used to indicate that information is collected from *all* the members of the group in which we are interested. The term survey (or *sample survey)* indicates that the information is collected only from some selected members of the group and not from all of them.

In addition to these two methods, we also come across regular collection and recording of information in a routine manner. The Meteorological Department maintains a daily rocord of maximum and minimum temperatures, rainfall, relative humidity, atmospheric pressure, etc. This information is used for weather forecasting. The Indian Railways keep a daily record of movement of passengers and goods, the income earned through fares and freight, etc. The Ministry of Commerce keeps a record of the quantity and value of goods imported and exported every month.

The information collected through censuses and surveys, or in a routine manner, is called *raw data*. The word *data* means information (its exact dictionary meaning is : given facts). The adjective raw attached to it indicates that the information thus collected and recorded cannot be put to any use immediately and directly but has to be *processed*, that is, converted to a more suitable form, before it begins to make sense to be utilised gainfully. Just as raw rice has to be cooked before it can be eaten and digested, the raw data too have to be converted into another form before any use can be made of it.

Take the All India Census, for example. The census enumerators go to every household and collect information about all the members. If we are interested in knowing how old the different persons alive at the time of the census were, our raw data coming from census records is simply a string of numbers: 11, 3, 42, 37, ... Each number stands for the age of a person, there being one number for each person alive at the time of the census. As the population of India exceeds 600 million our raw data is a string of more than 600 million numbers. In a sense this raw data has *all* the information about the ages of different persons on a given data. In another sense, it provides no information at all, because when we look at it we are simply bewildered by the very size of the data which has been collected. The Registrar's General office — a part of the census organisation — processes this raw data and presents it in form of different tables which make it easy to understand the information provided by the raw data. One such table is given below.

TABLE: 16.1

Table C-II : Age and Marital Status (Uttar Pradesh)

Age-Group	*Total* *Rural* *Urban*	*Persons*	*Male*	*Female*
All ages	T	88,311,144*	47,016,421	41,324,723
	R	75,952,548	40,214,012	35,738,536
	U	12,388,596	6,802,409	5,586,187
0-9	T	26,105,403	13,724,165	12,381,238
	R	22,626,315	11,914,955	10,711,360
	U	3,479,088	1,809,210	1,669,878

Age-Group	*Total Rural Urban*	*Persons*	*Male*	*Female*
10-14	T	10,859,933	6,061,626	4,798,307
	R	9,243,379	5,195,347	4,048,032
	U	1,616,554	866,279	750,275
15-19	T	7,184,548	3,978,135	3,206,413
	R	5,960,357	3,293,197	2,667,160
	U	1,224,191	784,938	539,253
20-24	T	6,531,468	3,246,939	3,284,529
	R	5,432,600	2,633,426	2,799,174
	U	1,098,868	613,513	485,355
25-29	T	6,474,870	3,275,180	3,199,690
	R	5,535,799	2,759,930	2,775,869
	U	939,071	515,250	423,821
30-34	T	5,910,572	3,013,133	2,997,439
	R	5,106,844	2,563,939	2,542,905
	U	803,728	449,194	154,534
35-39	T	5,174,221	2,719,620	2,454,601
	R	4,463,129	2,322,440	2,140,689
	U	711,092	397,180	313,912
40-44	T	4,737,880	2,550,573	2,187,307
	R	4,094,263	2,177,784	1,916,479
	U	643,617	372,789	270,828**
45-49	T	3,676,085	1,984,625	1,691,459
	R	3,186,065	1,693,346	1,492,719
	U	490,020	291,280	198,740
50-54	T	3,598,058	2,046,915	1,551,143
	R	3,144,675	1,772,218	1,369,457
	U	456,383	274,697	181,686
55-59	T	2,103,947	1,125,033	978,914
	R	1,853,821	978,649	875,172
	U	250,126	146,384	103,742
60-64	T	2,636,013	1,459,816	1,176,197
	R	2,336,025***	1,289,280	1,046,745
	U	299,988	170,536	129,452
65-69	T	1,241,876	681,421	560,455
	R	1,103,755	602,462	501,293
	U	138,121	78,959	59,162

Age-Group	Total Rural Urban	Persons	Male	Female
70+	T	2,099,009	1,145,516	953,493****
	R	1,861,574	1,013,535	848,039
	U	273,435	131,981	105,454
Age not stated	T	7,261	3,723	3,538
	R	6,947	3,504	3,443
	U	314	219	95

Source: Table C-II, p.48, Census of India 1971 (Uttar Pradesh), series 21, part II-C(ii)

* printed as 88,319,144

** printed as 270,823

*** printed as 2,136,025

**** printed as 953,423

This table gives the total population of Uttar Pradesh as 88,341,144 of which 47,016,421 (53.2%) are males and 41,324,723 (46.8%) are females. Nearly 86% (75,952,548/88, 341,144) of the population.lives in rural areas and only 14% are in urban areas.

The numbers of persons falling in different age-groups, that is, with age between 0 to 9 years, between 10 years and 14 years, and so on, are shown separately. At the end of the table we have the numbers of persons whose ages were not properly recorded at the time of the census. The total of such persons is 7,261, that is merely 0.008% of the whole population which shows the care and pains taken to record the information at the time of the census.

Compared to the raw data on ages, this table provides a very clear picture. We see,. after making some calculations, that 57,156,222 persons (64.7%) are below 30 years in age and 11,678,903 (13.2%) are 50 years or more in age. We can also obtain from the table the percentage of persons belonging to the different age-groups, that is the *age-distribution* of the population, the *sex-ratio* (ratio of females to males) in the different age-groups, and so on. For example, out of all the males $\frac{13,724,165}{47,016,421}$ = 29% are in the age-group 0—9, 13% in the age-group 10—14, and so on; the ratio of females to males in the age-group 30—34 is $\frac{2,542,905}{2,563,939}$ = 99% in rural areas, and 79% in urban areas, and so on.

The table as published by the census authorities had some printing errors. We have corrected the errors and given the corrected values in the table. The original values which were in error are given at the foot of the table. Can you see how these errors were discovered and corrected.

As another example, take a look at the table below which gives the death rates for males and females, separately and considered together, in different age-groups.

TABLE 16.2

Age-sex-specific death rates in rural India (1980). (Bihar and West Bengal are excluded) (per' 000 population)

Age-groups	Males	Females	Both
(1)	(2)	(3)	(4)
1-4	44.2	48.1	46.1
5-9	3.6	4.5	4.0
10-14	1.8	1.8	1.8
15-19	2.1	3.3	2.7
20-24	2.5	4.1	3.3
25-29	2.3	4.6	3.4
30-34	3.5	3.9	3.7
35-39	5.0	5.0	5.0
40-44	7.3	5.8	5.6(*)
45-49	9.7	7.7	8.8
50-54	15.1	10.9	13.1
55-59	22.0	17.3	19.8
60-64	35.9	28.4	32.2
65-69	62.2	43.2	52.6
70+	100.6	87.8	94.0
All ages	13.5	13.6	13.5

Source: Health Statistics of India, 1980 (taken from Table 8.14, Rural Development Statistics, NIRD, Hyderabad, 1985, pp.309-10)

These death rates are obtained by taking the ratio of the number of persons in a given age-group dying during a year to the total number of persons of that group alive at the beginning of the year. (The actual procedure is slightly different but the essential idea is the same). The ratio is not expressed as a percentage (i.e.,per 100 of the population above) but per 1000 persons alive. Thus the figure 44.2 at the top of column (2) states that out of 1000 males in the age-group 1-4 years alive at the beginning of the year, 44.2 had died by the time the year ended.Most demographic ratios (death rates, birth rates, etc.) are presented in this manner in terms of 1000 units in the denominator and not as percentages. The death rate 44.2 would, when expressed as a percentage, become 4.42.

Columns (2) and (3) give the death rates for males and females respectively, and column (4) gives the death rates for males and females taken together. Incidentally, there is some error in the values given in the row marked with an asterisk (*). Can you see why the values as given cannot be correct?

The raw data for this table would have consisted of a record of all persons alive at the beginning of the year with their sex and age, and a record of all deaths during the year with the sex and age at death.The raw data presented in the form of the table brings out certain interesting features which could not have been noticed by examining the raw data alone.

When we look at the male and female death rates for the whole population (in the row corresponding to "all ages") we find that the two are not very different. If we had only this information we could have concluded that the chances of dying during the year are the same for males and females. But when we look at the death rates for different age-groups (i.e., the *age-specific death rates)* we find that the reality is quite different. We find that the death rate for women is more than that of men in age-groups 1-4 and 5-9, and become equal for age-group 10-14. The death rate for women is again higher in age groups 15-19, 20-24, 25-29, 30-34 and then is lower than that for men in age-groups of 40 and above. Can you think of any reasons why it is so?

16.2 Variables of Observation

It we look at the Census example we find that for each person we have recorded the age and sex, and whether the person belongs to a rural or urban area. We say that we have taken observations on three *variables* : age, sex, and place of residence. The term variable thus stands for what is being observed. It is called a variable because the results obtained after observing it are different for different persons. Thus one person may have age 11, another 34, and so on. We say that 11, 34, ...are the *values* of the age-variable for these persons. Similarly, the values of the sex variable are "male" and "female". Now, the age variable can take any one of the values 1, 2, 3.... ., though we may not be able to *observe* all of them for there may be no person with age 3 in the population at a given time. Similarly, if there are only males in a group which is being observed, only the value "male" of the sex-variable will appear in our records. But when we think of the sex-variable as such we know that the value "female" can also occur. In view of this we make a distinction between the *observed values* and the *possible values* of a variable.

A variable is completely described by its descriptive name and the description of all the values it can possibly take. Usually only the name of the variable is given, its possible values being clear from the context.

In the census example we had another variable called "place of residence" and its possible values were taken as "rural" and "urban". The values of this variable could also have been recorded as the name of a State, district, city, village, etc. Thus two variables may have the same name but the sets of possible values may be different for them. We regard two variables as different if their sets of possible values are different, even if they have the same name.

16.3 Qualitative and Quantitative Variables

In the census example the values of the age-variable were numbers, those of the sex-variable were not numbers but the names "male" and "female" describing a certain type or quality. Similarly, the variable called "place of residence" also did not have numbers as its possible values, but the names "rural" and "urban".

Variables of observation with numbers as possible values are called *quantitative variables*: those with names of things, places, attributes, etc., as possible values are called *qualitative variables*. A word of caution is necessary here. The possible values of the sex-variable could have been *recorded* as 1 and 2 (1 standing for "male", 2 for "female"). The recorded values of this variable would then *appear* as numbers but that does not make the variable a quantitative variable. What is required for a variable to be called a quantitative variable is not the mere fact that its values are *recorded* as numbers, but that they are *really* numbers on which arithmetic operations can be meaningfully carried out. For example, it makes sense to talk of the sum, product and difference of two ages: but to talk of the sum of "male" and "female" values makes no sense even if they have been recorded as 1 and 2.

Age, height, area of a plot of land, weight of a basket of fruit, income of a worker, price of wheat, etc., are examples of quantitative variables. Examples of qualitative variables are sex, religion, caste, colour of hair, etc.

16.4 Units of Observation

We have explained above that a variable can take different values all of which may or may not be actually observed in a given situation. Each *recorded* or *observed* value of a variable is associated with a person, place, object, etc. Thus each recorded value of the age-variable refers to a particular person, each recorded value of land area refers to a particular plot of land, etc. Hence, in each case we have to clarify what each recorded value of a variable refers to or is associated with. The term *unit of observation* will be used to describe what the values of a variable are attached to. If we have the record of the results of a particular examination, the variable of observation could be the marks obtained (a quantitative variable) and the corresponding units of observation would be the students who had appeared in that examination. Each recorded value of the variable would then refer to a particular student.

In the census example, the units of observation are the persons alive at the time of the census and to each unit of observation we associate the value of three variables: age, sex, and place of residence. Thus different variables of observation may be associated with the same unit of observation.

16.5 Frequency Tables (or Frequency Distributions)

The frequency table is one of the important methods to present raw data in a form suitable for making the information contained in the raw data easily understandable.

We start with an example of a frequency table taken from the published results of the All India Agricultural Census conducted some years ago. We know that different persons own and cultivate different areas of land—some have more land than others, some have less. The Agricultural Census was carried out to obtain, besides other information, a fairly complete picture of the differences in size of land holdings in agriculture. For this purpose the Agricultural Census recorded the total land area which was in use wholly or partly for agricultural production and was operated as a single technical unit. Thus a holding of, say, one hectare in the Census records does not necessarily mean one piece of land (or a single field) measuring one hectare in area; it could also consist of more than one piece of land with the total area of all the pieces being one hectare. The raw data in this case would again be unintelligible but, presented in the form of the table below, it presents a very clear picture about the amount of agricultural land held by different persons.

TABLE 16.3

Number and area of holdings by size class of holdings, Uttar Pradesh (Plains)

Sl.No.	*Size-class (in hectares)*	*Total No.*	*Holdings Area*		
1	2	3	4	3(a)	4(a)
1.	Below 0.5	7116591	1521670	46.9	8.6
2.	0.5-1.0	2986638	2143859	19.7	12.1
3.	1.0-2.0	2591431	3642575	17.1	20.5
4.	2.0-3.0	1089301	2632107	7.2	14.8
5.	3.0-4.0	533765	1829241	3.5	10.3
6.	4.0-5.0	300480	1333858	2.0	7.5
7.	5.0-10.0	428585	2856156	2.8	16.1
8.	10.0-20.0	94727	1230970	0.6	6.9
9.	20.0-30.0	11752	278311	0.1	1.6
10.	30.0-40.0	3198	108131	0.02	0.6
11.	40.0-50.0	1058	46710	0.007	0.3
12.	50.0 and above	1406	138150	0.009	0.8
	Total	15158932	17761738	99.936	100.1

1. Columns 1 to 4 have been taken from Table III-1 Agricultural Census in Uttar Pradesh, 1970-71, Board of Revenue, U.P., Lucknow, p. 126.
2. Columns 3(a) and 4(a) are percentages obtained from columns 3 and 4 respectively.

In the second column of this table we have the values of the variable: "amount of land" divided into 12 *classes* (or groups). In the third column we have the number of farmers who were cultivating land with total area falling in a given class. Thus, 7,116,591 farmers cultivated land which was less than 0.5 hectares in area. Next came 2,986,638 farmers whose cultivated land had area between 0.5 and 1 hectare. Columns 2 and 3 of this table constitute a *frequency table*. We ignore the other columns for the present.

There are two steps in drawing up a frequency table from raw data. The first step is to use the *possible values of a variable of observation* (land area) to define a number of classes. The second step is to *count* the number of *units of observation* (farmers) for which the values of the variabe fall in a given class and to write down this number against the class. The number against each class is called the *frequency* of that class, and the total of all frequencies is called the *total frequency.* The total frequency is simply the number of *all* the units of observations for whom we have recorded the value of the variable of observation.

A frequency table can then be described by giving the variable and the units of observation. The description is briefly conveyed by the title. For example, the frequency table formed by columns 2 and 3. of the table from the Agricultural Census would have as its title: Frequency table *of* farmers *by* size of land cultivated.

The frequency table simply breaks up, or *distributes* the total frequency into the different classes of the table defined by means of the values of the variable of observation. For this reason we also use the term "frequency distribution" in place of "frequency table". The use of the word "table" draws attention to the form in which the raw data has been presented, whereas the word "distribution" emphasises that the total frequency has been divided into the frequencies of the different classes.

Care has to be taken while dividing the possible values of the variable into classes to form a frequency table to ensure that no possible value belongs to more than one class, and that every possible value belongs to some class or the other. If any value belongs to more than one class, the corresponding unit of observation would then be counted in the frequency of more than one class, so that the total frequency in the table would become *more* than the total number of units of observation. Again, if any value is left out, that is, does not belong to any class in the table, then a unit of observation having that value of the variable would not be counted in any of the class frequencies and the total frequency would be *less* than the total number of units of observation.

It should be kept in mind that the classes of a frequency table can consist of single values of the variable also; they need not always consist of more than one value.

If we look again at the frequency table of land holdings we find that the classes as stated appear to overlap with some values belonging to more than one class. Thus, the value 4.0 appears in the class 3.0 - 4.0, and also in the class 4.0 - 5.0. It would thus seem that the classes of the frequency table have not been defined correctly. However, a closer look at the table shows that this is not the case. The starting class has been defined as "less than 0.5 hectares", and the class at the end is defined as "50 hectares *and above*". We can thus conclude that the classes 0.5-1.0, 1.0-2.0.... etc., really stand for "0.5 hectares and above but less than 1.0 hectares", "1.0 hectares and above but less than 2.0 hectares", and so on. With this explanation we find that the classes of the frequency table have been defined properly.

As another example we construct a frequency table of the population of Uttar Pradesh by age from the Census Table given earlier. The division into classes by age is already given in it, the different classes being 0-9, 10-14, 15-19, ..., 65-69, 70 and above. Against each class we write the number of persons of that age from the column "persons" and in the row marked "T". The total population is 88,341,144. However, the ages of 7,261 persons are not known. Hence, we take the total frequency as 88,341,144 –7,261 = 88,333,883 to get the following frequency table.

TABLE 16.4

Frequency Table of Inhabitants of Uttar Pradesh by Age (1971 Census)

Age	*Frequency*	%
0-9	26,105,403	29.5
10-14	10,859,933	12.3
15-19	7,184,548	8.1
20-24	6,531,468	7.4
25-29	6,474,870	7.3
30-34	5,910,572	6.7
35-39	5,174,221	5.9
40-44	4,737,880	5.4
45-49	3,676,085	4.2
50-54	3,598,058	4.1
55-59	2,103,947	2.4
60-64	2,636,013	3.0
65-69	1,241,876	1.4
70 and above	2,099,009	2.4

The first two colunms constitute the frequency table. In the third column we have expressed the frequency of each class as a percentage of the total frequency. The usefulness of this column will be discussed later.

In this table too the division into classes does not seem to have been properly done. Though the classes do not overlap, that is, any two classes do not have a common value, yet some *possible* values of the variable are not included in any class. For example, the beginning class 0-9 ends at 9 and the next class starts with the value 10. Thus the ages between 9 and 10 years are in neither of the two classes. However, we know that in census records the age of person is recorded in *completed* years, so that, persons who have completed 9 years of age but are still below 10 years in age will be shown to have age 9 years. The class 0-9 therefore includes all people below 10 years of age and really stands for "Below 10 years". Similarly, the class 10-14 stands for "10 years and above but below 15 years". With this interpretation the frequency table is seen to have been correctly made.

It should have been clear by now that one may come across cases where the classes of a frequency table appear to overlap, or exclude some possible values of the variable. But the classes are seen to be properly defined if a certain rule of interpretation of the values beloniging to a class is followed. The rule is either stated clearly or, is evident from the context, or from the manner in which the values of the variable are recorded.

We can obtain many more frequency tables for the Census data table, such as

(i) frequency table of urban males by age
(ii) frequency table of inhabitants by place of residence
(iii) frequency table of rural inhabitants by sex.

Can you make these and other possible frequency tables yourself? Make note that while using sex or place of residence as the variable the total frequency will be the total number of persons in the Census Table. The number of persons in the category "Age not stated" at the end of the table will not be subtracted in this case.

16.6 Construction of Frequency Tables From Raw Data

The construction of a frequency table from given raw data is most conveniently done by the use of the method of tally marks. Suppose we have the following raw data giving the ages of 50 persons.

43, 11, 27, 2, 19, 63, 84, 9, 11, 31,
22, 19, 44, 43, 6, 14, 71, 58, 39, 27,
1, 25, 36, 64, 42, 51, 40, 18, 19, 17,
63, 40, 26, 37, 11, 16, 21, 47, 68, 70,
21, 22, 46, 35, 23, 14, 50, 48, 61, 5.

Let us divide the ages into the following classes: 0-9, 10-14, 15-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-(i.e., 70 or more).

The first column of our frequency table will show these groups. We now look at the first value recorded in our raw data. It is 43 and falls in the class 40-44. In the row corresponding to this class we make a mark "|" in the second column of the table entitled "tally marks". The second value in the raw data is 11 and for it we make the mark "|" in the row corresponding to the class 10-14. By the time we have covered the first 26 values of the raw data, ending at 51, there will be four tally marks for the class 40-44 corresponding to the values 43, 44, 43, 42. These four tally marks will appear as "||||" in our table. The 27th value in the raw data is 40 and it also gives a tally mark in the class 40-44. This fifth tally mark will be entered as "𝍸" by crossing diagonally the four tally marks already entered there. Thus every fifth tally mark in a class is entered by crossing diagonally the four tally marks preceding it. If there are 13 tally marks in all in a class they will appear as "𝍸 𝍸 |||". This method of recording tally marks makes counting the total number easier. The total number of tally marks of each class is than entered in the third column and gives the frequency of that class. Thus, our frequency table will finally look like this:

TABLE 16.5

Variable	*Tally Marks*	*Frequency*
0-9	𝍸	5
10-14	||||	4
15-19	𝍸 ||	7
20-24	𝍸	5
25-29	||||	4
30-34	|	1
35-39	||||	4
40-44	𝍸 |	6
45-49	|||	3
50-54	||	2
55-59	|	1
60-64	||||	4
65-69	|	1
70-	|||	3
Total	—	50

16.7 Relative Frequency Table

As we have explained earlier the frequency of any class in a frequency table is the number of units of observations for which the values of the variable belong to that class. Sometimes, this number (i.e., frequency) is expressed as a fraction of the total frequency and the fraction so obtained, usually expressed as a percentage, is called the *relative frequency* of the class. The relative frequencies in a frequency table make it easier to understand and assimilate the information, particularly, when the class frequencies are large.

For example, if we look at the frequency table No. 16.4 in which the numbers of persons belonging to different age-groups are given, the individual class frequencies are so large that we cannot make comparisons between different age-groups easily. If, however, we look at the relative frequencies we see that almost a third (29.5%) of the population is below 10 years in age. We also notice that persons of age 65 years or more constitute only about one-tenth (2.4 + 3.0 + 1.4 + 2.4= 9. 2%) of the total population. It is also clear that the number of persons of age 35 or more (28.8%) is nearly equal to the number of children of age less than 10 years, and this fact cannot be noticed so easily if one looks at the actual frequencies.

Similarly, if we look at the relative frequencies in the frequency table of land-holdings given in column 3(a) of the Table No.16.3 certain interesting features stand out much more clearly as compared to the table of actual frequencies. For example, though there are persons owning more than 50 hectares of land their number is very small, just about .01%. Majority of persons own small pieces of land; nearly half (46.9%) own less than half a hectare of land, and two-thirds (46.9 + 19.7 = 66.6%) own less than one hectare of land.

You may have noticed that the total of relative frequencies, which should be exactly 100 in theory, is more than 100 (= 100.1) in the census example, and less than 100 (= 99.936) in the land-holdings example. This is really not as surprising as it may appear at first; the difference from the theoretical value of 100 arises due to the fact that percentages given there are not exact but close approximations of the exact value. In the census frequency table the percentage 29.5 stands for

$$\frac{26105403}{88333883} \times 100 = 29.553\ldots$$

and is thus *less* than the true value. Similarly, for the age-group 40-44 the relative frequency 5.4% stands for

$$\frac{4737880}{88333883} \times 100 = 5.363$$

and, therefore, *more* than the true value. While calculating the percentages we have *rounded off* the, percentages due to which some percentages are more, and some less, than the exact percentage so that the total of all percentages will sometimes be more, and sometimes less, than 100. Can this total be sometimes equal to 100 also?

Relative frequencies are also very useful when we have to compare two or more frequency distributions. For example, suppose the ages of the inhabitants of two villages result in the following frequency distributions, where the relative frequencies are also shown.

TABLE NO. 16.6

Age-group	No. of inhabitants		Relative frequency (%)	
	Village A	Village B	Village A	Village B
0-5	18	35	12.6	15.3
5-10	26	38	18.2	16.7
10-20	25	40	17.5	17.5
20-30	29	32	20.3	14.0
30-40	21	27	14.7	11.8
40-50	14	23	9.8	10.1
50-60	6	18	4.2	7.9
60-80	4	15	2.8	6.6
Total	143	228	100.1	99.9

The relative frequencies express the frequency of any class as a percentage of the total frequency. Thus for village A, the relative frequency of the 0-5 class is

$\frac{18}{143} \times 100 = 12.6$, while for village B it is $\frac{35}{228} \times 100 = 15.3$

When we look at the frequencies of the different age-groups of the two villages, we cannot easily see if the two are similar or different. We cannot, for example, find out easily if one village consists mostly of younger or older people as compared to the other. When we examine the relative frequencies, we can quickly discern some interesting features. Village *B* has a higher proportion of children in age-group 0-5. If we look at age-groups 5-10 and 10-20 taken together, we find that the proportion of inhabitants of ages from 5 to 20 years (18.2 + 17.5 = 35.7%) in village *A* is almost the same as the proportion of such persons (16.7 + 17.5 = 34.2%) in village *B* As we go down the relative frequency table, we see that village *A* has a higher proportion of persons with ages from 20 to 40 years than village *B*. the proportions for them being 35.0% (20.3 + 14.7) and 25.8% (14.0 + 11.8) respectively. Consequently, the proportion of older persons with age ranging from 40 to 80 years is more in village *B* (10.1 + 7.9 + 6.6 = 24.6%) than in village *A* (9.8 + 4.2 + 2.8 = 16.8%)

It must be noted that for comparing two frequency distributions by means of the relative frequencies, the variable as well as the division of the values of the variable into classes, must be the same for both distributions.

16.8 Graphical Presentation of Frequency Distributions

The main features of a frequency distribution are conveniently communicated by

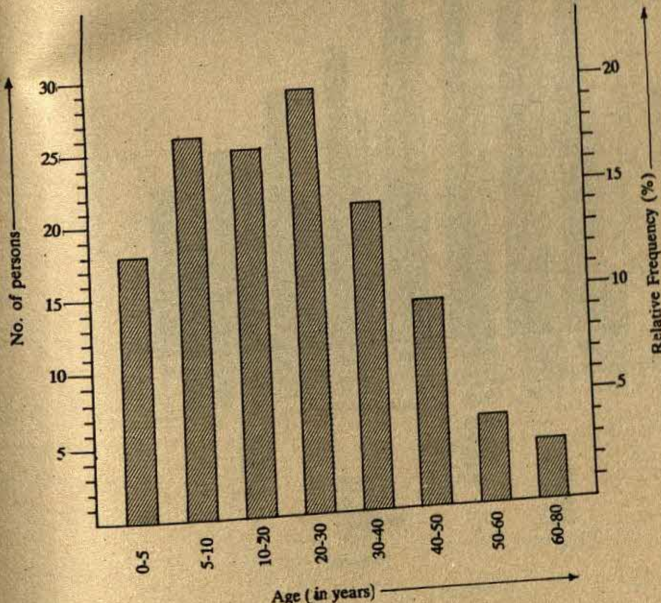


Fig. 16.1

representing the frequency distribution in the form of a diagram, since a diagram more easily and more quickly understood than a collection of numbers. Diagrammatic presentation is particularly useful when the number of classes or the clas frequentcies in a frequency distribution are large.

There are various methods of graphical presentation of frequency distribution which are in use. We shall only discuss two of them, viz., the *bar diagram* and th *pie diagram.*

Consider the example of the frequency distribution by age of inhabitants of villag *A* which we introduced while discussing the use of the conçept of relative frequency The *bar diagram* for this frequency distribution is shown below.

To draw the bar diagram we mark equal lengths for the different classes on th horizontal axis. Notice that the same length is used for every class even thoug different classes have different age ranges. Some span a period of 5 years, some 1 and the last one spans a period of 20 years. On each of these lengths on the horizonta axis we erect a rectangle whose height is *proportional* to the frequency of the clas represented by the base. We thus get a number of "bars" which give the diagram it name. We could also make the heights of the rectangles proportional to the relativ

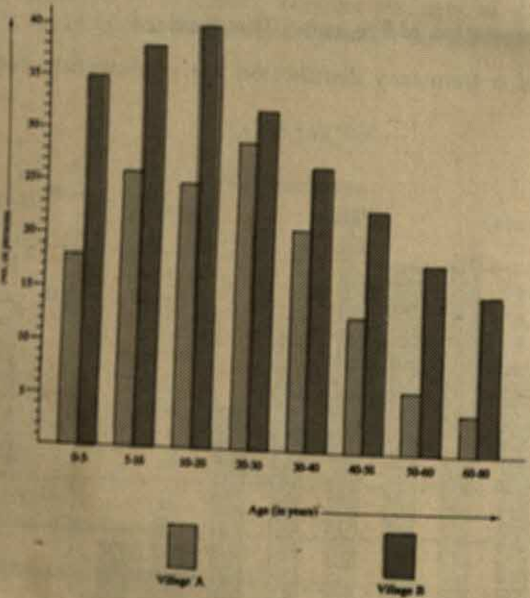


Fig. 14.2

frequencies to get the bar diagram of the relative frequency distribution. In fact, only a single bar diagram has to be drawn which serves both for the frequency distribution as well as the relative frequency distribution. The scale on the vertical axis indicates whether the bar diagram represents the frequencies or the relative frequencies. In the figure above we have shown the scale for the frequencies on the left, and for the relative frequencies on the right.

If we have two frequency distributions with the same classes,we can represent both on the same bar diagram. For example, if we have the distributions by age of village *A* and village *B* in the example mentioned above, we can represent the two frequency distributions in the same diagram as shown below. The vertical scale for the frequencies is shown on the left.

However, we cannot represent the two frequency distributions as well as their relative frequency distributions in a single diagram as we were able to do for a single frequency distribution. If we are interested in comparing graphically two frequency distributions with the same classes,we have to represent their relative frequency distributions in a single diagram. This has been done for our example in the diagram below in which the vertical scale now represents the relative frequency percentages.

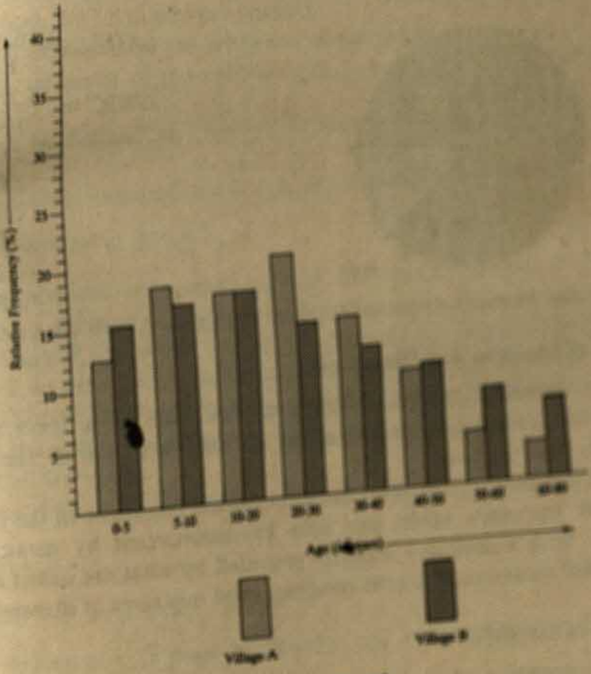


Fig. 10.3

In the *pie diagram*, which should be used for representing relative frequency distributions only, the relative frequencies are represented by *sectors* of a circle. We start with a circle of arbitrary radius and then draw radii from the centre to the circumference. In this way the circle is divided into as many parts as there are classes in a frequency distribution. The *area* of each sector is proportional to the relative frequency of the class represented by the sector.

We take again the example of village A. There are eight classes, and the relative frequencies (expressed as percentages) are 12.6, 18.2, 17.5, 20.3, 14.7, 9.8, 4.2 and 2.8. The *pie diagram* for this relative frequency distribution is shown below:

We start by drawing a radius OA. This is followed by drawing the radii OB, OC. so that the area of the sector AOB is proportional to 12.6, that of sector BOC is proportional to 18.2, and so on, till all the radii are drawn to divide the area of the circle into eight sectors. We know from our knowledge of geometry of the circle that the area of any sector is proportional to the angle subtended at the centre by its arc. Hence, what we require is that the angles AOB, BOC be proportional to 12.6, 18.2,... respectively. Since, the total of all the eight angles at the centre is 360°, the eight angles are determined by dividing 360° into eight parts in the proportion 12.6 : 18.2 : 17.5 : ...4.2 : 2.8. Hence,

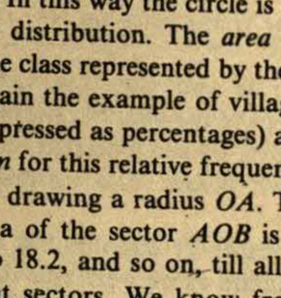


Fig. 16.4

$$\angle AOB = \frac{12.6 \times 360}{100.1} = 45.3^\circ$$

$$\angle BOC = \frac{18.2 \times 360}{100.1} = 65.4^\circ$$

$$\angle COD = 62.9^\circ$$

$$\angle DOF = 73.0^\circ$$

$$\angle EOF = 52.8^\circ$$

$$\angle FOG = 35.2^\circ$$

$$\angle GOH = 15.1^\circ$$

$$\angle HOA = 10.0^\circ$$

The pie diagram can be easily drawn when these angles have been obtained.

16.9 Measures of Location and Dispersion

So far we have discussed the presentation of raw data in a form suitable for communicating the information contained in it, and have studied the use of the frequency table for this purpose.

In the case of *quantitative variables* the information contained in the raw data, or in the associated frequency table, can also be *summarised* by means of a few numerical values. Such a summary is partly provided by what are called *measures of location* (also called *measures of central tendency)* and *measures of dispersion.*

16.10 Measures of Location

The two most commonly used measures of location are the *arithmetic mean* (or the *mean* in short) and the *median*. The mean and the median lie between the largest and the smallest of the observations, and are, therefore, also called measures of central tendency. Both can be said to lie, in some sense, at the centre of the observations. But the precise meaning of the centre is, as we shall see, different for the mean and the median.

(a) *The Arithmetic Mean*

The arithmetic mean of a given set of observations is simply the *average* of all the values of the variable recorded in the raw data. The raw data from which we constructed the frequency table by the method of tally marks consisted of the ages of 100 persons. The mean of these 100 values is given by

$$\frac{43+11+27+\ldots+48+61+5}{100}$$

$$=\frac{1686}{100}=16.86$$

We say that the *mean age* of the 100 persons is 16.86 *years per person*. Note that we do not simply say that the mean age (or mean) is 16.86 but also mention the unit of measurement in which the quantitative variable has been measured (here it is years) as well as the units of observation (here persons) from whom the observations have come. In the same way, if the average marks of a class of students equal 47.5, we will say that the mean is 47.5 marks per student.

The general definition of the arithmetic mean is as follows:

The arithmetic mean of the values $x_1, x_2, \ldots, x_n$ of a variable recorded for n units of observation is defined as $\frac{x_1+\ldots+x_n}{n}$ per unit of observation. The expression $x_1+\ldots+x_n$ is usually denoted by $\sum_{i=1}^{n} x_i$ (or simply Σx_i), and the mean, denoted usually by $\bar{x}$, is written as $\bar{x}=\frac{1}{n}\sum_{i=1}^{n} x_i$

The mean provides a summary of the information contained in the raw data $x_i \ldots, x_n$. Being a summary it does not communicate all the information in the raw data. If we know $\bar{x}$, we cannot say anything about the actual values $x_1, x_2, \ldots, x_n$. But, if with $\bar{x}$ we also know the value of n, we can obtain the total of all the values $x_1, \ldots, x_n$, which from the definition of $\bar{x}$ can be seen to be equal to $n\bar{x}$. We also know that some values are more and some less than $\bar{x}$. Thus, in some sense, $\bar{x}$ lies at the centre of all the values.

From the definition of $\bar{x}$ we get

$$\sum_{i=}^{n} (x_i-\bar{x})=0 \text{ i.e.}$$

$$(x_1-\bar{x})+(x_2-x)+\ldots+(x_n-\bar{x})=0$$

Thus, some values $x_i - \bar{x}$ must be positive and some negative, otherwise the sum would not be zero.

If we add all the positive $x_i - \bar{x}$ and all the negative $x_i - \bar{x}$, these two sums will have the same *magnitude* but will have different signs, so that their algebraic sum is zero. When we say that $\bar{x}$ lies at the centre of all the observations, we are simply saying that the sums of the positive $x_i - \bar{x}$ and the negative $x_i - \bar{x}$ have the same magnitude, and the algebraic sum is zero. In this case being at the centre does not imply that half the observations are more than $\bar{x}$ and half are less than it. For, it can happen that $x_i - \bar{x}$ is positive for a small number of observations and negative for a large number, or vice-versa.

Calculation of Arithmetic Mean from Frequency Tables: Suppose we have the following frequency table of families by the number of children in which each class of the frequency table is defined by a *single* value of the variable of observation.

TABLE 16.7

No. of children	*Frequency*
0	43
1	55
2	60
3	64
4	48
5	34
Total	304

The table tells us that the number of children in each of the 304 families being studied was recorded, and it was found that 43 families had no children, 55 had only 1 child, and so on. There are thus 304 values of the variable which have been observed and the value 1 occurs 55 times, 2 occurs 60 times, and so on. To get the mean of these 304 values, we have first to add all of them. That sum is most conveniently obtained by using multiplication and is given by

$$(0 \times 43) + (1 \times 55) + (2 \times 60) + (3 \times 64) + (4 \times 48) + (5 \times 34) = 729$$

Thus the mean is $\frac{729}{304} = 2.4$ (approx.) children per family.

In general, if the values in the different classes of the frequency table are $x_1, \ldots, x_m$ and the frequencies are $f_1, \ldots, f_m$, the mean will be given by

$$\bar{x} = \frac{\sum_{i=1}^{m} f_i x_i}{\sum_{i=1}^{m} f_i}.$$

Consider next the frequency table of farmers by the area of land held, given earlier. Here the classes are no longer defined by single values; each class consists of many values of the variable. In such a case, we cannot determine the *exact* value of the mean because all the values of the variable are not known exactly. So, what one tries to do is to get an *approximate* value of the mean. The method of approximation is to replace all the observed values belonging to a class by the *mid-value* of that class and then use the mid-value to determine the mean. In this particular example, we are faced with a difficulty as the last class is defined as "50 hectares and above" so that its mid-value cannot be obtained. So we ignore this class, and use the remaining classes only to illustrate the method.

The calculations are presented as follows by adding two more columns to the frequency table for the mid-value and the product of the mid-value and the frequency.

TABLE 16.8

Variable	*Frequency* f	*Mid-value* x	$f.x$
Below 0.5	7,116,591	0.25	1,779,147.75
0.5 — 1.0	2,986,638	0.75	2,239,978.50
1.0 — 2.0	2,591,431	1.50	3,887,146.50
2.0 — 3.0	1,089,301	2.50	2,723,252.50
3.0 — 4.0	533,765	3.50	1,868,177.50
4.0 — 5.0	300,480	4.50	1,352,160.00
5.0 — 10.0	428,585	7.50	3,214,387.50
10.0 — 20.0	94,727	15.00	1,420,905.00
20.0 — 30.0	11,752	25.00	293,800.00
30.0 — 40.0	3,198	35.00	111,930.00
40.0 — 50.0	1,058	45.00	47,610.00
Total	15,157,526		18,938,495.25

What we have done is to treat each class as defined by a single value, the mid-value of that class, and then followed the method applicable to such frequency tables. The approxmiate value of the mean is given by

$$x = \frac{18,938,495.25}{15,157,526}$$

= 1.249 hectares per cultivating unit.

= 1.25 hectares (approx.)

EXERCISE 16.1

1. Prove that $(x_1 - \bar{x}) + (x_2 - \bar{x}) + ... + (x_n - x) = 0$

where $\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n}$

2. (a) The marks obtained by 20 students in a test were
   13, 17, 11, 5, 18, 16, 11, 14, 13, 12, 18, 11, 9, 6, 8, 17, 21, 22, 7, 6

   Find the mean marks per student.

   (b) If extra 5 marks are given to each student, show that the mean marks are also increased by 5.

   (c) If 2 marks are deducted for each student, show that the mean marks are also reduced by 2. (See exercise 3 for the reason).
3. If x is the mean of $x_1, x_2, \ldots, x_n$, show that the mean of $x_1 + a, x_2 + a, \ldots, x_n + a$ is $\bar{x} + a$, where a is any number positive or negative.
4. (a) If all the marks in 2(*a*) are doubled, show that the mean value is also doubled.

   (b) The postal expenses on the letters despatched from an office on a given day resulted in the following frequency distribution:

Postage (p)	*No. of letters*
15	47
30	33
35	56
60	41
70	25

   Find the mean postage per letter. Convert the postal charges in rupees and then calculate the mean postage per letter.
5. If x is the mean of $x_1, x_2, \ldots, x_n$, show that the mean of $ax_1, ax_2, ax_2, \ldots, ax_n$ is ax, where a is any number different from zero.
6. The mean age (in years) per student and the number of students in each of the four classes of two primary schools are given below:

	School A		*School B*	
	No.	*Mean age*	*No.*	*Mean age*
Class I	6	6.2	25	7.1
Class II	10	7.5	32	8.4
Class III	28	8.6	12	9.2
Class IV	30	10.0	4	10.7

   (a) Obtain the mean ages per student for the two schools.

   (b) In each class, the mean age of students of school *A* is less than that of students in the same class of school *B*. But we find that the mean age per student for the whole school is more for school *A* than for school *B*. Why?

7. The ages of all the male inhabitants of a village were recorded and the following frequency table was obtained.

Age (Years)	*No. of persons*
0—5	12
5—10	18
10—20	16
20—30	19
30—40	14
40—50	11
50—60	4
60—80	3

Obtain the mean age per male inhabitant.

8. The measurements (in mm) of the diameters of the heads of 107 screws gave the following frequency table:

Diameter	*Frequency*
33—35	17
36—38	19
39—41	23
42—44	21
45—47	27

Calculate the mean head diameter per screw.

(b) The Median

The median is also a value lying at the centre of all the observations but in a different sense as compared to the mean. In the case of the median the number of values which are more than the median equals the number of values which are less than it. For example, suppose the values of the variable are

15, 5, 7, 3, 11, 28, 9, 43, 28, 17, 31

Then five values (3, 5, 7, 9, 11) are less than 15, and five values (17, 28, 28, 31, 43) are more than it. Thus we can say that 15 is the *median* (or *median value*) of the given set of values.

In the above example, the number of values was 11, an odd number, so that one could obtain 5 values more and 5 less than the value at the centre. Let us see what happens when the total number of observations is an even number. For example let the values be

15, 5, 7, 3, 11, 16, 28, 9, 43, 28, 17, 31.

Suppose we count for each value, starting with the smallest, the number of values which are more than it and the number of values which are less. We obtain the following table

TABLE 16.9

Value (x)	3	5	7	9	11	15	16	17	28	31	45
No. of values x	0	1	2	3	3	5	6	7	8	10	11
No. of values x	11	10	9	8	7	6	5	4	2	1	0

In this case, we do not find any value which is such that the number of values more than it *equals* the number of values less than it. One could however agree to call the values 15 and 16 as coming nearest to the definition of median. Also, if we take the number 15.4, which is not an observed value, it could be called the median as it has exactly 6 values more and 6 less than it. But then *any* number between 15 and 16, say 15.6 or 15.7, could also serve as the median of the observation. Looked at in this way one finds that the median may not be a single vlaue or number, but can be one of a set of values.

Another problem is created when a large number of values are repeated. Consider the eleven observations (arranged in ascending order)

3,5,7,8,15,15,15,15,15,15,43.

We have 11 observations and we cannot pick any value, or even any number as the median in the sense that it will have equal number of values above and below it.

The general idea of the median as a measure of central tendency is easy to understand. We are looking for a value, or a number, which divides the given observed values into two parts, half the number of observations being more, and half less, than the median value. The median is not only a value lying at the centre of the observations but also conveys very useful information. Thus, if we say that the median monthly income of a group of workers is Rs. 500.00 then it is clear that half the workers earn more, and half less than Rs. 500.00 per month. But its calculation poses some difficulties as we have seen from the examples considered by us. These difficulties are overcome by a slight modification in the definition which is as followes:

"The median of a given set of n values of a quantitative variable is *any number* M such that

(i) [No. of values $\geq M$] $\geq \frac{n}{2}$ (ii) [No. of values $\geq M$] $\geq \frac{n}{2}$

Let us apply this definition to the three examples considered above.

In the first example, the values when arranged in increasing order are

3, 5, 7, 9, 11, 15, 17, 28, 28, 31, 43.

Here $n = 11$, $\frac{n}{2} = 5.5$. For a number M to be the median, we require

(i) (No. of observations $\geq$ M) ≥ 6

(ii) (No. of observations $\leq$ M) ≥ 6

To satisfy (i), we count 6 observations from the right and see that we must have $M \leq 15$. Similarly, counting 6 observations from the left, we see that we must have $M \geq 15$. Hence, $M = 15$.

In the second example, the observations arranged in increasing order are

3, 5, 7, 9, 11, 15, 17, 28, 28, 31, 43.

Here $n = 12$, $\frac{n}{2} = 6$. Hence, counting 6 observations from the right, we see that $M \leq 16$. Similarly, counting 6 observations from the left, we see that, $M \geq 15$. Thus 15 and 16 and any number lying between them can be taken as the median according to our definition.

Applying the same procedure to the third example, we find that 15 is the median for the set of observations

3, 5, 7, 8, 15, 15, 15, 15, 15, 43.

Here $\frac{n}{2} = 5$, and the 5th observations from the right and from the left are both equal to 15. Hence, $M = 15$.

The use of the general definition determines the median without ambiguity. However, in some cases the median may not be unique. In such cases we follow a *convention* (that is, a procedure or rule accepted by everyone) according to which the average of the smallest and largest median values is taken as the median. In the second example, the smallest median value was 15 and the largest median value was 16. So, by our convention we take 15.5 as the median in this case.

Calculation of the Median: If the observations are given as raw data, we arrange them in increasing (or decreasing) order and count the total number of observations. If the number n of observations is odd, say, $n = 2m+1$, then $\frac{n}{2} = m + \frac{1}{2}$. Hence, from the definition of median, we find, as in the first example above, that the $(m + 1)$th observation is the median. For example, if there are 13 observations, we have $13 = 2 \times 6 + 1$, so that $m = 6$. Hence, in this case the 7th observation will be the median.

If n is even, say, $n = 2m$, we have $\frac{n}{2} = m$. Then, as in the second example above, the median will be any number between the mth observation from the right and the mth observation from the left. The average of these two observations is then taken as the median. In other words, we take the average of the mth and the $(m+1)$ th value as the median.

If the observarions are given in the form of a frequency table with classes defined through single values, we rearrange the frequency table so that the values of the variable appear in ascending or descending order. We then add a column of *cumulative frequencies,* the cumulatlive frequency of any class being the total of the frequencies of that class and all classes coming before it in the table. If the total frequencies is odd ($= 2m + 1$), we locate,by means of the cumulative frequencies,the class in which the $(m + 1)$th observation falls and the value defining that class is the median. If the total frequency is even ($= 2m$), we locate similarly the classes and then the values corresponding to the mth and $(m + 1)$th observations and take their average as the median.

As an example, we consider the frequency table of families by number of children, which we uesd to illustrate the method of calculating the arithmetic mean. (See Table 16.7) The total frequency is 304, an even number. As $304 = 2 \times 152$, we have to locate the 152nd and 153rd observations. For that we look at the column of cumulative frequencies in the table given below:

TABLE 16.10 152th

No. of Children	*Frequency*	*Cumulative Frequency*
0	43	43
1	55	98
2	60	158
3	64	222
4	48	270
5	34	304
Total	304	—

From the column of comulative frequencies we notice that the first 43 observations are all equal to 0, all the observations from the 44th to the 98th are equal to 1, all from the 99th to the 158th are equal to 2 and so on. Thus, the 152 th and 153 th observations are both equal to 2 so that the median number of children per family is 2. What it suggests is that of the families observed half had less and half more than 2 children.

Consider now the case where each class of the frequency table consists of more than one vlaue. As in the case of the mean, we cannot now calculate the exact value of the median and have to go in for an approximation. The approximation method, however, is different from that which was adopted for the calculation of the mean.

We first arrange the classes of the frequency table in ascending or descending order of the vlaues of the variable. Then we locate, with the help of the cumulative frequencies, the class in which the median value lies. However, since we are calculating an approximate value only, we make no distinction between an odd or an even number of obsevations. If n is the total number of observations, we obtain the value of $\frac{n}{2}$ and then locate the class in which the median lies as the class for which the cumulative frequency first equals or exceeds the value $\frac{n}{2}$.

Having located the class in which the median lies, called the *median class,* we now use our approximation method to find the actual median vlaue in that class. For approximation, we assume that all the observations of that class are uniformly spread over the whole class. Thus, if the class consists of values between 15 and 25 and there are 20 observations falling in that class, we *assume* that these 20 observations are so spread that the diference between two consecutive observations is $\frac{25-15}{20} = 0.5$. Thus, the 20 observations can be taken as 15.5, 16.0, 16.5, 17.0, 17.5, 18.0, 18.5, 19.0, 19.5, 20.0, ..., 24.5, 25.0. The median value inside the median class is then determined on this assumption.

As an example, consider the frequency table of farmers by the amount of land

TABLE 16.11

Variable	*Frequency*	*Cumulative Frequency*
Below 0.5	7,116,591	7,116,591
0.5 – 1.0	2,986,638	10,103,229
1.0 – 2.0	2,591,431	12,694,660
2.0 – 3.0	1,089,301	13,783,961
3.0 – 4.0	533,765	14,317,726
4.0 – 5.0	300,480	14,618,206
5.0 – 10.0	428,585	15,046,791
10.0 – 20.0	94,727	15,141,518
20.0 – 30.0	11,752	15,153,270
30.0 – 40.0	3,198	15,156,468
40.0 – 50.0	1,058	15,157,526
50.0 or more	1,406	15,158,932
Total	15,158,932	

owned which we have already come across. To the column of frequencies, we add the column of cumulative frequencies. No rearrangement of the classes is needed as they are already arranged according to ascending values of the variable. Notice that we have not excluded the class "50 hectares and above" on the use of mid-values of the classes.

Here $n = 15{,}158{,}932$, so that $\frac{n}{2} = 7{,}579{,}466$ which is less than the cumulative frequency of the "0.5 – 1.0" class and more than the cumulative frequency of the "Below 0.5" class. Hence, the median class is the "0.5 – 1.0" class and the median value lies between 0.5 and 1.0 hectare. To find the median, we have to locate the 7,579,466th observation in this class. The 7,116,591th observation can be taken to be 0.5 hectares. The next 2, 986, 638 observations equally spaced between 0.5 and 1.0, that is they cover a range of $1.0 - 0.5 = 0.5$ hectares. To reach the 7,579,466th observation, we have to find the $7{,}579{,}466 - 7{,}116{,}591 = 462875$th observation in the median class. The first 462875 observations in the median class will, according to our assumption to get an approximate value of the median, cover a range of

$$\frac{0.5}{2986638} \times 462875 = 0.08 \text{ hectares.}$$

Hence, the median value is given by

$$0.5 + 0.08 = 0.58 \text{ hectares.}$$

We thus see that even though there are farmers owning 10, 20, or even more than 50 hectares of land, nearly half the farmers own less than 0.58 hectares of land.

The general expression for finding the median in such cases is given by

$$M = l + \frac{\frac{n}{2} - f_0}{f_1} \cdot h$$

where

M is the median,
l is the lower value of the median class,
f_0 is the cumulative frequency of the class just before the median class,
f_1 is the frequency of the median class,
h is the range (upper value — lower value) of the median class,
n is the total frequency.

It is assumed that the Classes have been arranged according to ascending values of the variable.

EXERCISE 16.2

1. The number of students absent in a school was recorded every day for 147 days

and the raw data was presented in the form of the frequency table below:

No. of students absent	*No. of days*
5	1
6	5
7	11
8	14
9	16
10	13
11	10
12	70
13	4
15	1
18	1
20	1
	Total 147

Obtain the median and describe what information it conveys.

2. The marks obtained out of 50 by 102 students in a test were according to the frequency table below:

Marks	*Frequency*
20	8
22	15
23	28
24	27
26	20
31	2
38	1
43	1

Obtain the median and describe what information it conveys.

3. The following table gives the frequency distribution of married women by age at marriage:

Age (in years)	*Frequency*
15-19	53
20-24	140
25-29	98
30-34	32
35-39	12
40-44	9
45-49	5
50-54	3
55-59	3
60 and above	2

Calculate the median and interpret the result.

4. In the example of land holdings we found that the mean was 1.25 hectares and the median only 0.58 hectares. What do you conclude from it?

16.11 Measures of Dispersion

Supposing you were the captain of a cricket team and had selected the whole team except for one more batsman. Two batsmen, whom we shall call A and B, were available and you had to choose one of them to complete the team. The runs scored by these two batsmen in the last 5 matches (10 innings) were as follows.

A: 38, 70; 48, 34; 42,55; 63,46; 54,44 (Total: 494)
B: 5, 11; 8,29; 83, 104; 20,28; 81, 123 (Total: 492)

The average score per innings is nearly 50 for both the batsmen. So which one do you select? If your team already has batsmen on whom you can depend, you could select B in the hope that he will hit up a big score. Of course, you would then be taking the risk, that he may get dismissed on a poor score. If you do not want to take such a risk you would select A since his scores do not change much from innings to innings and you can be more certain that he would make a reasonably good score. Though the two batsmen have almost the same average score per innings, A's scores do not show much variation from innings to innings while B's scores show great variation with very high scores in some innings and very low scores in some.

If we plot the scores of A and B on a straight line, we see that the points corresponding to the scores of A are close to each other, or bunched together, and those corresponding to the scores of B are scattered or spread out.

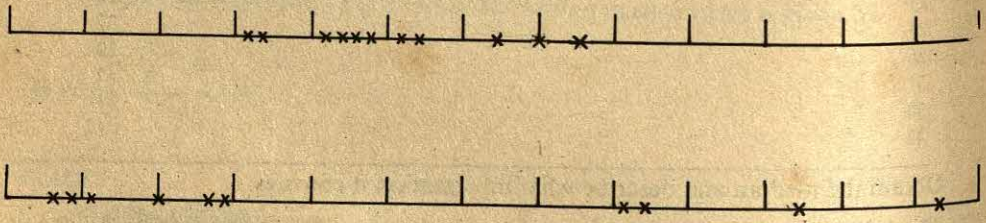


Fig. 16.5

We use the term dispersion to indicate this scattering or spreading out of the different values of a quantitative variable. Thus, we will say that the scores of B show a higher dispersion than the scores of A.

Let us again look at the scores of A and B. The mean score per innings of A is 49.4, and his scores are close to the mean score, the difference between the scores and the mean score ranges from $34 - 49.4 = -15.4$ to $70 - 49.4 = 20.6$. The mean score for B is 49.2, but his scores in the different innings are not close to the mean value but located away from it as compared to A. In his case the difference between the scores and the mean value ranges from $49.2 - 5 = 44.2$ to $123 - 49.2 = 73.8$. We say that A's scores have a smaller *dispersion about the mean* as compared to B's scores.

We have already discussed the use of measures of location to *summarise* the information provided by the values of a quantitative variable. The measures of location, which we have studied, the mean and the median, give us a sort of central value around which the values of the variable are located, some values being more and some less than the central value. But they do not give us any idea of how far these values are from the central value. The measures of dispersion are designed to provide this information, that is to indicate if the different values of the variable are close to a measure of location or away from it. A *measure of dispersion about the mean* will tell us if the observations are close to the mean value or not. Similarly, a *measure of dispersion about the median* will tell us about the degree of scatter of the observations about the median.

The commonly used measure of dispersion about the mean is the *variance* or, the *standard deviation* which is simply the square root of the variance. The *measure of dispersion about the median*, which we shall study, is the *mean deviation.*

A small value of the variance will indicate that most of the observations do not differ very much from the mean, and variance will go on increasing in value as the observations get further and further away from the mean value. Similarly, small values of the mean deviation will indicate that the observations are close to the median whereas large values will indicate that the observations differ much from the median.

The two measures of location, the mean and the median, provide a very inadequate summary of the information conveyed by a set of observations, but the mean along with the variance (or standard deviation), and the median along with the mean deviation provide a summary which is adequate in most situations.

(a) *Variance and standard deviation*

Consider the scores of batsmen *A*. If we calculate the differences of these scores with their mean value, we get $38 - 49.4 = -11.4$, $70 - 49.4 = 20.6$, and so on.The ten differences are -11.4, 20.6, -1.4, -15.4, -7.4, 5.6, 13.6, -3.4, 4.6, -5.4. Similarly, calculations for *B* give the ten differences equal to -44.2, -38.2, -41.2, -20.2, 33.8, 54.8, -29.2, -21.2, 31.8, 73.8. Comparison of *these differences show that they are on the whole* larger for *B* as compared to the differences for *A*. And because of that, *B's* scores have a higher dispersion about the mean in comparison to *A*. We will see below that the definition of variance is based upon this idea.

Suppose our observations are $x_1, \ldots, x_n$ and their arithmetic mean is $\bar{x}$. Let us now look at the differences $x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x}$. If these differences are small *in magnitude* it means that the values $x_1 \ldots, x_n$ are close to the mean value $\bar{x}$, if they are large in magnitude it implies that the values are scattered away from the mean value. So, a measure of the degree of scatter around $\bar{x}$, that is a measure of dispersion around $\bar{x}$ could be constructed by utilising the differences $x_1 - \bar{x}, \ldots, x_n - \bar{x}$. If we simply add these differences, some of which will have a positive value and some a negative value, we will get the sum equal to zero. So, the sum of the differences cannot be used to obtain an idea of the total dispersion of the values about the mean $\bar{x}$. The difficulty is due to the fact that some differences are positive and some negative. We overcome this difficulty by *squaring* all the differences and then adding them to get the non-negative sum

$$(x_1-\bar{x})^2+\dots+(x_n-\bar{x})^2=\Sigma\,(x_i-\bar{x})^2$$

If this sum is zero, each difference $x_i - \bar{x}$ must be zero so that there is no dispersion at all as all the observations are equal to the mean value $\bar{x}$. If this sum is small, all the individual differences will be small in magnitude, indicating that the values $x_1, \dots, x_n$ are all close to the mean $\bar{x}$. If this sum is large, then most of the individual differences will be large in magnitude (though some may be small) thus indicating a higher degree of dispersion about $\bar{x}$.

Thus the sum $\sum_{i=1}^{n}(x_i-\bar{x})^2$ is a reasonable indicator of the amount of dispersion about the mean value $\bar{x}$. But there is another difficulty to be overcome. It can happen that we have a large number of observations for which the individual differences $x_i-\bar{x}$ are small in magnitude so that we should get a small value for the measure of dispersion. But, it can happen that the sum $\sum_{i=1}^{n}(x_i-\bar{x})^2$ is larger in this case than when we have very few observations with comparatively larger values for the individual differences. We would then be led to conclude that the dispersion is more in the first case than in the second even though the situation is quite the opposite.

For example, the 5 numbers 10, 20, 30, 40, 50 have mean equal to 30 so that $\Sigma(x_i-\bar{x})^2$ in this case is equal to

$$(-20)^2-(-10)^2+(10)^2+(20)^2=1000.$$

And the 41 numbers
20, 20.5, 21, 21.5, 22, 22.5, 23, 23.5, 24, 24.5, 25, 25.5,
26, 26.5,27, 27.5, 28, 28.5, 29, 29.5, 30, 30.5, 31, 31.5,
32, 32.5, 33, 33.5, 34, 34.5, 35, 35.5, 36, 36.5, 37, 37.5,
38, 38.5, 39, 39.5, 40
also have mean equal to 30 with $\Sigma(x_i-\bar{x})^2$ equal to

$$(-10)^2+(-2.5)^2+(-9)^2+\dots+(-0.5)^2+(0.5)^2+\dots+(9.5)^2+(10)^2=1435.$$

Thus, if we use $\Sigma(x_i-\bar{x})^2$ as our measure of dispersion about the mean, we would say that the second set of numbers has a higher dispersion about the mean than the first. Obviously, such a conclusion would be wrong, as all the individual differences $x_i-\bar{x}$ between -10 and 10 in the second case while in the first the individual differences $x_i-\bar{x}$ range from -20 to 20. So, in effect the observations in the first case are not as close to the mean as in the second and should, therefore, give a higher value for the dispersion.

The difficulty is created by the fact that the total number of observations in the two cases is different. In the second case, though the values $x_i-\bar{x}$ are small, the total $\Sigma(x_i-\bar{x})^2$ for 41 observations becomes larger than the similar total for the first case in which there are only 5 observations.

We overcome this difficulty by dividing the total $\Sigma(x_i - \bar{x})^2$ by the number of observations. The resulting value $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is called the *variance* of the set of observations $x_1, x_2, \ldots, x_n$. With this modification, the dispersion of the first set of observations is measured by $\frac{1000}{5} = 200$, and that of the second set of observations by $\frac{1435}{41} = 35$. And we now see that as expected the first set has higher dispersion than the second.

The exact definition of the variance is as follows:

"The variance of a given set of observations $x_1, x_2, \ldots, x_n$, of a quantitative variable is the number value

$$\frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \ldots + (x_n-\bar{x})^2}{n}$$

where $\bar{x}$ denotes the arithmetic mean of the observations."

The symbol σ^2 is normally used for the variance. We shall sometimes write the variance of the observations $x_1, x_2, \ldots, x_n$ as $\sigma^2(x_1, x_2, \ldots, x_n)$ if we wish to draw attention to the fact that we are talking of the variance of the n observations $x_1, x_1, \ldots, x_n$.

The variance is a measure of dispersion about the mean value $\bar{x}$; large values of the variance indicating greater dispersion than small values. However, the variance, being based on the squares of the differences $x_i - \bar{x}$ will not be measured in the same units as x_i and $\bar{x}$. If, for example, the observations x_i, and hence the mean $\bar{x}$, are lengths measured in metres, the variance will be measured in square metres, a unit of area and not of length. For this reason, the dispersion about the mean is usually expressed in the form of the square root of the variance which is called the *standard deviation* of the set of observations. The standard deviation is usually denoted by the symbol σ (to show its relationship with the variance σ^2), or, in short by s.d.

The exact definition of the standard deviation is as follows:

"The *standard deviation* of a given set of observations $x_1, x_2, \ldots, x_n$ is the number given by

$$+\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

where $\bar{x}$ is the arithmetic mean of the observations." A square root has two values, one positive and one negative (+2 and −2 are both square roots of 4). For the standard deviation, we always take the positive value of the square root of the variance as indicated by the sign (+) in front of the square root sign in the definition above.

If we have a frequency table with m classes with each class defined by a *single* value x_i with frequency f_i, the variance will be defined by

$$\sigma^2 = \frac{\sum_{i=1}^{m} f_i(x_i - \bar{x})^2}{\sum_{i=1}^{m} f_i}$$

where

$$\bar{x} = \frac{\sum_{i=1}^{m} f_i x_i}{\sum_{i=1}^{m} f_i}$$

If each class of the frequency table consists not of a single value but a number of values, we follow the approximation procedure which was followed for the calculation of the mean. Each class is replaced by a single value equal to the mid-value for that class and then the above procedure for single value classes is followed to get an *approximate* value of the variance.

Calculation of Variance: For the actual calculation of the variance, we do not use the formulae given in the definition. Instead we make use of the following *identity*:

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - \frac{(\Sigma x_i)^2}{n}$$

to obtain the value of $\sum_{i=1}^{n} (x_i - \bar{x})^2$ by means of the expression $\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$. This has the advantage of reducing the number of arithmetic operations to be carried out (for example, we have one subtraction only in place of n subtractions), and of decreasing the chance of committing errors of calculation. It also reduces rounding-off errors.

So, for calculating the variance we shall use the formula

$$\sigma^2 (x_1, x_2, \cdots, x_n) = \frac{1}{n}\left[\sum_{i=1}^{n} x_i^2 - \frac{(\Sigma x_i)^2}{n}\right]$$

In case of frequency tables with m classes, each class being defined by a single value x_i with frequency f_i, the variance will be calculated by the formula

$$\sigma^2 = \frac{\sum_{i=1}^{m} f_i x_i^2 - \frac{(\Sigma f_i x_i)^2}{\Sigma f_i}}{\Sigma f_i}$$

The same formula will be used to get the approximate value of the variance if the classes of the frequency table consist not of single values but of a number of values. We shall use the mid-value x_i of the class with frequency f_i in the above formula.

Example 16.1

The scores of batsman A were 38, 70, 48, 34, 42, 55, 63, 46, 54, 44.

The total being 494,we have $\bar{x} = 49.4$. The differences $x_i - \bar{x}$ are

$$-11.4, 20.6, -1.4, -15.4, -7.4, 5.6, 13.6, -3.4, 4.6, -5.4.$$

Hence,

$$\Sigma (x_i - \bar{x})^2 = (11.4)^2 + (20.6)^2 + \cdots + (4.6)^2 + (5.4)^2$$
$$= 1126.4$$

and

$$\sigma^2 = \frac{1126.4}{10} = 112.64$$
$$\text{s.d.} = \sqrt{112.64} = 10.61$$

We have calculated the variance by using the definition of the variance. We now show how to obtain it by the formula for calculating it. In that formula,we have first of all to obtain the sums Σx_i and Σx_i^2 (this is done in the table below), and then these sums are used to calculate the variance .

	x_i	x_i^2
	38	1444
	70	4900
	48	2304
	34	1156
	42	1764
	55	3025
	63	3969
	46	2116
	54	2916
	44	1936
Total	494	25530

$$\frac{(\Sigma x_i)^2}{n} \quad \frac{(494)^2}{10}$$
$$= 24403.6$$
$$\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$$
$$= 25530 - 24403.6 = 1126.4$$
$$\sigma^2 = \frac{1126.4}{10} = 112.64$$

Hence, s.d. = 10.61

Example 16.2

The scores (x_i)obtained by 21 children in an intelligence test are given along with their frequencies (f_i) in the first two columns of the table below. The next two columns are used to obtain the values of $f_i x_i$ and $f_i x_i^2$. The totals of these columns are then used to calculate the variance as shown. Note that the values $f_i x_i$ in the third column are obtained by multiplying the values x_i and f_i in the first two columns. The value of $f_i x_i^2$ is then obtained by multiplying the value of $f_i x_i$ in the third column by the value x_i in the first column. In this way, the squaring of the values x_i is avoided and calculation time saved.

x_i	f_i	f_ix_i	$f_ix_i^2$
91	3	91x3 = 273	273x91 = 24843
92	2	184	16928
96	3	288	27648
97	2	194	18818
101	5	505	51005
103	3	309	31827
108	3	324	34992
Total	21	2077	206061

$$\frac{(\Sigma f_ix_i)^2}{\Sigma f_i} = \frac{(2077)^2}{21}$$

$$= \frac{4313929}{21} = 205425.19$$

$$\Sigma f_ix_i^2 - \frac{(\Sigma f_ix_i)^2}{(\Sigma f_i)} = 206061 - 205425.19 = 3635.81$$

$$\sigma^2 = \frac{3635.81}{21} = 173.1338$$

Hence, s.d. $= \sigma = 13.16$

Example 16.3

The measurements (in mm) of the diameters of the heads of 107 screws gave the frequency table formed by the first two columns below. The third column gives the mid-value of the class from which the remaining columns as well as the value of the variance are obtained as in Example 16.2.

Diameter	*Frequency*	*Mid-value*	f_ix_i	f_ix^2
33-35	17	34	578	19652
36-38	19	37	703	26011
39-41	23	40	920	36800
42-44	21	43	903	38829
45-47	27	46	1242	57132
Total	107		4346	178424

$$\frac{(\Sigma f_i x_i)^2}{\Sigma f_i} = \frac{(4346)^2}{107} = 176520.71$$

$$\Sigma f_i x_i^2 - \frac{(\Sigma f_i x_i)^2}{\Sigma f_i} = 178424 - 176520.71$$

$$= 1903.29$$

$$\sigma^2 = \frac{1903.29}{107} = 17.79$$

Hence, s.d. $= \sigma = 4.22$

Short-Cut Method for $\bar{x}$ and σ^2

If the values of x_i or the mid-values x_i in the different classes of a frequency table are large, the calculations of $\bar{x}$ and σ^2 become quite lengthy and time-consuming. In the case of frequency table with equally spaced values or mid-values x_i, calculations can be simplified to a great extent by a simple method described as a 'Short-cut Method'. According to this method we first subtract a constant A from each x_i; A is generally chosen to be the middle x_i or a value near to the middle x_i in the frequency table. Each deviation $x_i - A$ is then divided by a suitable constant h which is generally the class-interval in the frequency table.

Taking

$$u_i = \frac{x_i - A}{h} \text{ or } x_i = A + hu_i,$$

we have

$$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma x_i} = \frac{\Sigma f_i (A + hu_i)}{\Sigma f_i}$$

$$\therefore \quad \bar{x} = A + h\,\frac{\Sigma f_i u_i}{\Sigma f_i}$$

$$\text{and } \sigma^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{\Sigma f_i}$$

$$= \frac{\Sigma f_i (A + hu_i - A - h\bar{u})^2}{\Sigma f_i}$$

$$\therefore \quad \sigma^2 = \frac{h^2}{\Sigma f_i}\left(\Sigma f_i u_i^2 - \frac{(\Sigma f_i u_i)^2}{\Sigma f_i}\right)$$

Let us work out below the Example 16.3 by using the above formula for x and σ^2:

Diameter	*Frequency* f_i	*Mid-Value* x_i	$u_i = \frac{x_i - 40}{3}$	$f_i u_i$	$f_i u_i^2$
33-35	17	34	−2	−34	68
36-38	19	37	−1	−19	19
39-41	23	40	0	0	0
42-44	21	43	+1	+21	21
45-47	27	46	+2	+54	108
Total	107			+22	216

Here $A = 40$ and $h = 3$

$$\bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i} = \frac{22}{107} = 0.21$$

$$\bar{x} = 40 + 3 \times 0.21 = 40.63$$

$$\frac{(\Sigma f_i u_i)^2}{\Sigma f_i} = \frac{(22)^2}{107} = 4.52$$

$$\Sigma f_i u_i - \frac{(\Sigma f_i u_i)^2}{\Sigma f_i} = 216 - 4.52 = 211.48$$

$$\therefore \quad \sigma^2 = \frac{9 \times 211.48}{107} = 17.79$$

Hence, s.d. $= \sigma = 4.22$.

EXERCISE 16.3

1. Prove the identity

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}$$

2. (a) What would be the variance of the scores of batsman A if he had scored 10 runs more in each innings?

(b) Calculate the variance of the scores of batsman A given that his scores in the 10 innings were $38 - 10 = 28$, $70 - 10 = 60$, $48 - 10 = 38$; $34 - 10 = 24$; $42 - 10 = 32$; $55 - 10 = 45$; $63 - 10 = 53$; $46 - 10 = 36$; $54 - 10 = 44$; $44 - 10 = 34$.

3. Suppose the observations $x_1, x_2, \ldots\ldots x_n$ are changed to $x_1+a, x_2+a, \ldots, x_n+a$, where a is a given positive or negative number. Show that the variance remains unchanged.
[*Hint:* The mean x becomes $x+a$]

4. (a) The scores of 10 students in a test, in which the maximum marks were 50, were as follows:
28, 36, 34, 28, 48, 22, 35, 27, 19, 41. Find the variance.
(b) Later on the maximum marks were increased to 100, and accordingly each student's score was doubled. Find the variance of the new scores.

5. Given that σ^2 is the variance of the observations $x_1, x_2, \ldots, x_n$, prove that the variance of $ax_1, ax_2, \ldots, ax_n$, where a is *any* number different from zero, is a^2.

[Hint: $\bar{x}$ becomes $a\bar{x}$ and $x_i - \bar{x}$ becomes $a(x_i - \bar{x})$]

6. The scores of 48 children in an intelligence test are shown in the frequency table below:

Score	Frequency
71	4
76	3
79	4
83	5
86	6
89	5
92	4
97	4
101	3
103	3
107	3
110	2
114	2

Calculate the variance σ^2 and find out the percentage of children whose scores lie between $\bar{x} - \sigma$ and $\bar{x} + \sigma$.

(b) *Mean Deviation about the Median*

Let us now consider the problem of measuring the dispersion about the median. Once again we begin with the individual differences $x_i - M$ between the observations x_i and the median M. If the observations are close to the median, these differences will be small in magnitude. If on the other hand the observations are far from the median,they will be large. Since M is the median of the obervations x_i, half of the differences x_i-M will be positive and half negative. Because of it,we cannot just take the average of these values as a measure of dispersion. We had a similar problem in the case of the variance where we squared each difference $x_i-\bar{x}$ to make all of them positive. Now we follow a slightly different method. We simply ignore the negative sign if x_i-M is negative. In other words, we replace each difference $x_i - M$ by its *absolute value* $|x_i - M|$. Thus if $x_i - M$ is 3.5,we have $|x_i - M| = 3.5$ and if $x_i - M$ *is equal to* $- 4.01$, we take $|x_i - M| = 4.01$. We add these absolute values to get the sum

$$|x_1-M| + |x_2 - M| + \ldots + |x_n - M| = \sum_{i=1}^{n} |x_i - M|$$

which is an indicator of the amount of dispersion of the observed values about the median. This sum is then divided by the number of observations n to get the mean deviation about the median. The division by n is done for the same reason for which $\sum_{i=1}^{n} (x_i - \bar{x})^2$ was divided by n to get the variance as a measure of dispersion about the mean.

The definition of the *mean deviation about the median* (or just *mean deviation* in short) is as follows:

"The mean deviation of a given set of observations $x_1, x_2, \ldots, x_n$, about their median M is the number

$$\frac{|x_1 - M| + |x_2 - M| + \ldots + |x_n - M|}{n}."$$

The individual difference in,absolute value, $|x_i - M|$ are called *deviations* of the observations from M and the mean deviation, as the name suggests, is their average. Since, unlike the variance, the mean deviation is measured in the same units as the observations, no further modifications are needed and the mean deviation can be used as it is as a measure of dispersion about the median.

We could also have measured the dispersion about M by obtaining the average

$$\frac{(x_1 - M)^2 + (x_2 - M)^2 + \ldots + (x_n - M)^2}{n}$$

and then taking its square root. Similarly, the dispersion about the mean $\bar{x}$ could have been measured by the average

$$\frac{|x_1 - \bar{x}| + |x_2 - \bar{x}| + |x_n - \bar{x}|}{n}$$

However, we follow different methods while measuring dispersion about the mean $\bar{x}$ and about the median M. The reasons for doing so cannot be explained at this stage and will be learnt by you when you study the subject at a more advanced level. For the present it is enough to remember that the variance (or standard deviation) is used to measure the dispersion about the mean, and the mean deviation is used to measrue the dispersion about the median.

Calclation of the mean Deviation : Suppose the observations are $x_1, x_2, \ldots, x_n$, and their median value is M, If we calculate the mean deviation according to its definition,we have to obtain the values of $x_i - M$ for all the observations. However, this is not necessary and we can do the calculations more easily and in a shorter time as follows:

(i) Separate the observations x_i which are $\geq M$ from those x_i which are $< M$.
(ii) Count the number (n_1) of the observations which are $\geq M$, and obtain their total (x_1).
(iii) Count the number (n_2) of the observations which are $< M$, and obtain their total (x_2).
(iv) Check that $n_1 + n_2 = n$.
(v) Obtain the value of the mean deviation about the median as

$$\frac{(x_1 - x_2) - (n_1 - n_2)M}{n_1 + n_2}$$

If the observations are given in the form of a frequency table with m classes in which each class consists of a single value x_i with corresponding frequency f_i, we proceed as follows:

(i) Take the observations x_i which are $\geq M$; obtain the sum (n_1) of their frequencies f_i and the sum (x_1) of the products $f_i x_i$ for them.
(ii) Take the observations x_i which are $< M$;obtain the sum (n_2) of their frequencies f_i, and the sum (x_2) of the products $f_i x_i$ for them.
(iii) Check that $n_1 + n_2$ equals the total frequency.
(iv) Obtain the mean deviation as

$$\frac{(x_1 - x_2) - (n_1 - n_2)M}{n_1 + n_2}$$

Example 16.4

We consider the scores

38, 70, 48, 34, 42, 55, 63, 46, 54, 44 in ten innings of batsman A. If we rearrange the scores in increasing order, we get

34, 38, 42, 44, 46, 48, 54, 55, 63, 70. The fifth value is 46 and the sixth is 48. Hence,we take their average 47 as the median M.

With this value of M, the mean deivation according to the definition is given by

$$\frac{1}{10}[\,|34 - 47| + |38 - 47| + \ldots + |70 - 47|\,]$$
$$= \frac{1}{10}[13 + 9 + 5 + 3 + 1 + 1 + 7 + 8 + 16 + 23]$$
$$= \frac{1}{10}[86] = 8.6$$

If we use the alternative method of calculating the mean deviation, we find that

(i) observations 48, 54, 55, 63, 70 are $\geq M$ so that $n_1 = 5$ and $x_1 = 290$

(ii) observations 34, 38, 42, 44, 46 are $< M$ so that $n_2 = 5$ and $x_2 = 204$.

Hence, mean deviation is given by

$$\frac{(290-204)-(5-5).47}{5+5} = \frac{86}{10} = 8.6$$

In this particular example, we have $n_1 = n_2$ so that the term $(n_1 - n_2)\,M$ was reduced to zero. However, we need not necesarily have $n_1 = n_2$ as will be seen in Example 16.5.

Example 16.5

Suppose the observations when arranged in increasing order are:

3, 3, 5, 9, 10, 12, 12, 12, 18, 21, 21, for which median $M = 12$. We now have

$x_1 = 12 + 12 + 12 + 18 + 21 + 21 = 96$

$x_2 = 3 + 3 + 5 + 9 + 10 = 30$

$n_1 = 6$

$n_2 = 5.$

Thus, we have mean deviation

$$= \frac{(96-30)-(6-5)12}{6+5} = \frac{66-12}{11} = \frac{54}{11}$$

If we follow the definition, we get the mean deviation as

$$\frac{1}{11}\Big[|3-12| + |3-12| + |5-12| + |9-12| + |10-12| + |12-12| +$$

$$|12-12| + |12-12| + |18-12| + |21-12| + |21-12|\Big]$$

$$= \frac{1}{11}[9+9+7+3+2+0+0+0+6+9+9] = \frac{54}{11}$$

Exercise 16.4

1. The lengths (in cm) of 30 small pieces of cloth left after a day's sale in a cloth shop were:
42.7, 98.5, 29.3, 29.1, 89.2, 69.2, 70.8, 5.7, 22.8, 49.1, 44.3, 79.5, 66.4, 14.2, 10.6, 41.1, 91.8, 86.4, 49.4, 85.5, 56.1, 49.6, 72.8, 59.4, 6.4, 32.9, 62.6, 37.3, 56.0, 33.1
Find the mean deviation d about the median M. See how many are between $M - 2d$ and $M + 2d$.

2. The areas (in square meters) of the different fields in a village were as follows: 67.1, 74.5, 94.9, 98.2, 15.3, 91.5, 80.2, 42.3, 62.0, 75.3, 57.0, 88.0, 49.4, 34.1, 56.1, 31.2, 36.1, 83.2, 37.8, 80.3, 85.6, 53.9, 69.9, 71.9, 91.5, 98.3, 42.2, 32.6, 91.6, 40.8, 35.5.

   Find the absolute mean deviation d about the median M. See how many observations are between $M - d$ and $M + d$, and how many (in per cent) are between $M - 2d$ and $M + 2d$.

CHAPTER 17

LINEAR PROGRAMMING

17.1 **Linear Constraints**

Suppose we have a sum of Rs 35 with us with which we wish to purchase pencils which cost Rs 2 each, and notebooks which cost Rs 3 each. How many pencils and note books can we buy? If we buy pencils only, we can buy at most 17 pencils and will have Re 1 left; if we buy notebooks only, we can buy at most 11 note-books and will have Rs 2 left with us. We can also buy 5 pencils and 8 note-books at a total cost of Rs $[(5 \times 2) + (8 \times 3] =$ Rs 34 and will have Re 1 left, or, we can buy 7 pencils and 7 note-books, in which case we will have spent the total amount of Rs 35 which we had with us. Obviously, we cannot buy 8 pencils and 7 note-books since their total cost is Rs 37 which is more than what we can spend. We thus see that we have many choices regarding the numbers of pencils and notebooks we can purchase without requiring more than Rs 35.

Let us denote the number of pencils by x, and that of note-books by y. Obviously, x and y can take only positive and integral values. The problem we have been discussing can be put in mathematical language as follows:
Find positive integers x and y such that

$$2x + 3y \leq 35.$$

The solutions to this problem, that is all possible pairs (x, y) which satisfy the above conditions can be graphically presented if we use our knowledge of coordinate geometry. If we draw the line $2x + 3y = 35$, then we know that on one side of the line we have points for which $2x + 3y > 35$ and on the other side we have points for which $2x + 3y < 35$.
The two regions are shown below:

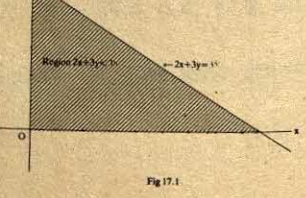


Fig 17.1

Since in our problem, we want x and y to be positive integers (x and y can be zero but cannot be negative). we see that the solutions to our problem are just those points (x,y) whose coordinates are integers and which fall in the shaded region OAB (including the boundaries) in the figure below.

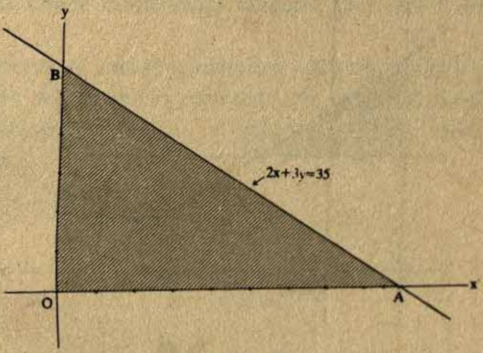


Fig 17.2

If we buy pencils only, the solution, will be located on OA and if we buy notebooks only, the solution will lie on OB. If we are looking for a solution which completely utilises the money available (e. g., $x = 4$, $y = 9$; $x = 1$, $y = 11$; $x = 7$, $y = 7$, $x = 10$, $y = 5$, etc.), the solutions will be found on the line segment AB.

If we have to buy a minimum of 2 pencils and 3 notebooks, then our choice of x and y should be such that we also have $x \geq 2$ and $y \geq 3$, in addition to the requirement $2x + 3y \leq 35$. The points (x,y) for which y is non-negative and $x \geq 2$, and for which x is non-negative and $v \geq 3$ are shown below.

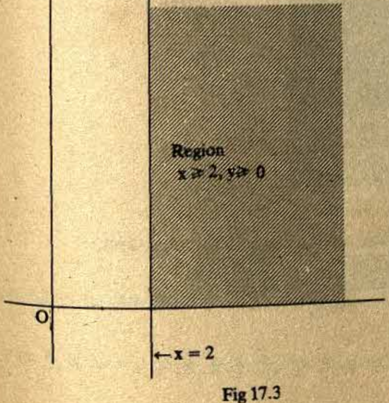


Fig 17.3

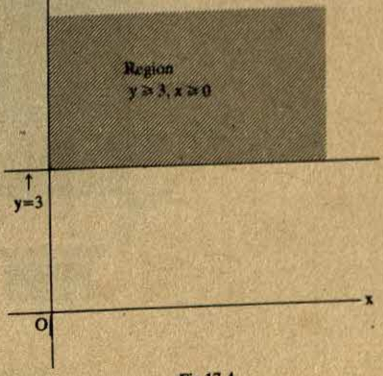


Fig 17.4

With this added restriction that we should have $x \geq 2$ and $y \geq 3$, the solutions now are pairs of integers for which the point (x,y) lies in the shaded region given below,

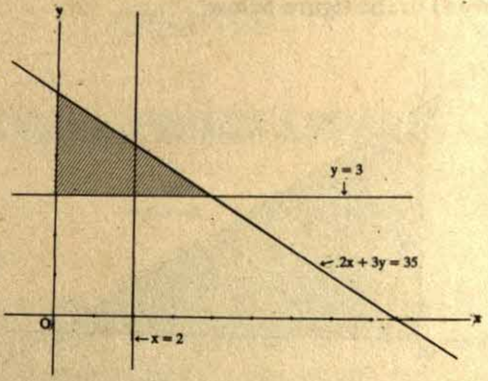


Fig. 17.5

which is bounded by the lines $2x + 3y = 35$, $x = 2$, and $y = 3$.

It can happen at times that the conditions we have laid down are such that they cannot be satisfied. In such cases there will be no solution to the problem. For example, suppose we wish to purchase at least 3 pencils and 4 notebooks and do not want to spend more than Rs 15. Clearly, this is impossible. The geometrical representation of these conditions is shown below.

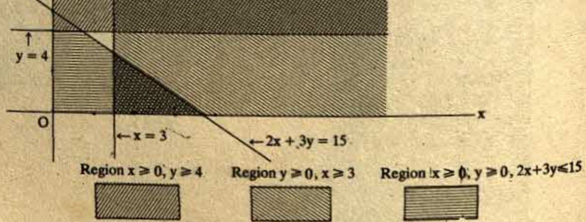


Fig 17.6

We see that there is no region where all the conditions are satisfied. In such cases we say that there is no solution, or that the solution set is *empty.*

The conditions $2x + 3y \leq 5$, $x \geq 3$, etc. are also called *constraints* because they restrict our freedom of choice of the values x and y. The constraints we have used are of the type called *linear constraints* because they are expressed by means of *linear functions.*

An important property of linear constraints on two variables x and y is that the set of points (x, y) for which the constraints are satisfied is either empty, or is a region bounded by straight lines (i.e. a polygon), or an unbounded region with straight line boundaries.

Example 17.1

The constraints

$$x + y \leq 5$$
$$4x + y \geq 4$$
$$x + 5y \geq 5$$
$$x \leq 4$$
$$y \leq 3$$

have as solution set the shaded area (a polygon) in the figure below:

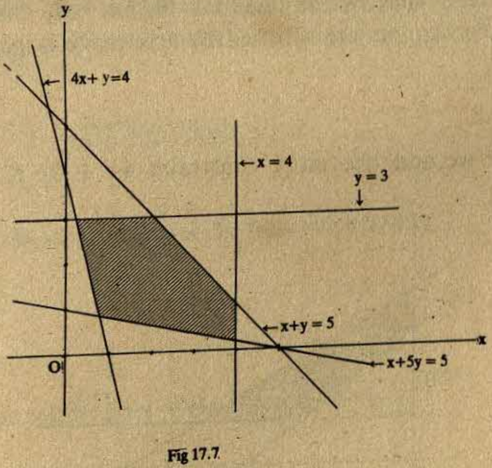


Fig 17.7

This area is bounded by the five lines

$x + y = 5$, $4x + y = 4$, $x + 5y = 5$, $x = 4$, and $y = 3$.

Example 17.2

The solution set of the constraints

$$3x+4y \geq 12$$
$$y \geq 1$$
$$x \geq 0$$

is the shaded portion shown in the figure below:

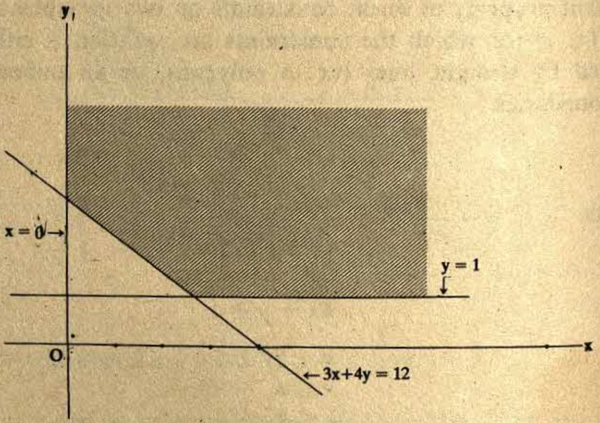


Fig 17.8.

It is bounded on three sides by tne lines $x = 0$, $3x + 4y = 12$ and $y = 1$, but is unbounded as the constraints are satisfied by arbitrarily large positive values of x and y.

Example 17.3

If in example 17.2 we add one more constraint $4x + 7y \leq 28$. The solution set

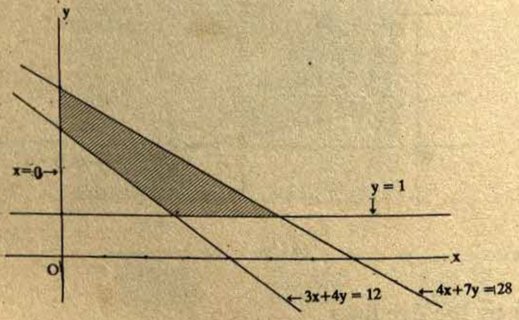


Fig 17.9

(shaded area in the figure above) is no longer unbounded, but is a quadrilateral (i.e., a polygon) bounded by the lines $x = 0$, $3x + 4y = 12$, $y = 1$ and $4x + 7y = 28$.

Example 17.4

This time if we add to example 17.2 the constraint $x + 2y \leq 3$, the solution set is mpty as is clear from the figure below:

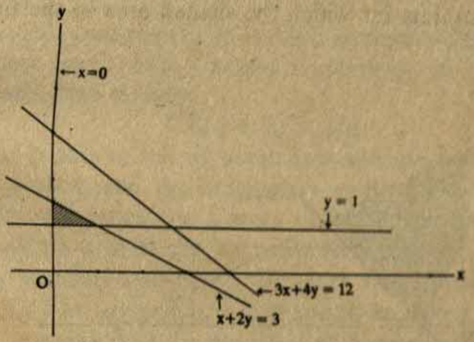


Fig 17.10

The shaded area in example 17.2 satisfies the three constraints but does not atisfy the new constraint. The shaded area of the figure above satisfies $x \geq 0$, ≥ 1 and $x + 2y \leq 3$, but it does not satisfy the remaining constraint $3x+4y \geq 12$.

EXERCISE 17.1

1. Draw the diagram of the solution set of the linear constraints
$2x + 3y \leq 6$
$x + 4y \leq 4$
$x \geq 0$
$y \geq 0$

2. Exhibit graphically the solution set of the linear constraints
$x + y \geq 1$
$y \leq 5$
$x \leq 6$
$7x + 9y \leq 63$
$x, y \geq 0$

3. Verify that the solution set of the following linear constraints is empty:

$x - 2y \geq 0$
$2x - y \leq -2$
$x \geq 0$
$y \geq 0$

4. Verify that the solution set of the first two constraints of problem No. 3 above is not empty, and that it is unbounded.

5. Find the linear constraints for which the shaded area in the figure below is the solution set.

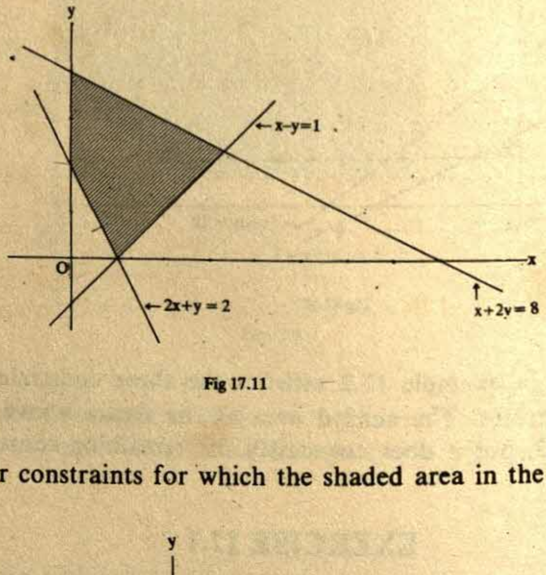


Fig 17.11

6. Find the linear constraints for which the shaded area in the figure below is the solution set.

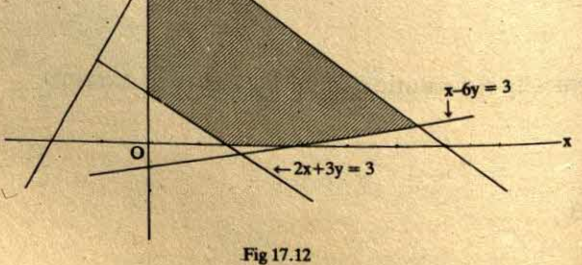


Fig 17.12

17.2 LINEAR PROGRAMMING

We now look at the problem of purchase of note-books and pencils from the point of view of a shopkeeper who buys goods in order to sell them at a profit. The number of pencils and note-books he can purchase will depend on the amount of money he can invest. He has also to make sure that he does not buy more than what he can sell. Finally, he will try to see that the number of pencils and note-books which he purchases is such as to give him the maximum amount of profit on sale.

Suppose the pencils cost him Re 0.75 each, and the note-books Rs 3 each. If he has Rs 100 to invest, and x and y denote respectively the numbers of pencils and note-books purchased, then we have

$$0.75\,x + 3y \leq 100.$$

Now suppose he is able to sell all the pencils and note-books purchased by him, he pencils at Re 1 each, and the note-books at Rs 3.50 each. Thus he makes a rofit of Rs 0.25 on each pencil and a profit of Re 0.50 on each note-book. Since he makes a larger profit on note-books, we could be tempted to suggest that the shopeeper should only purchase note-books. However, that would not be correct advice. or, by investing Rs 3 in the purchase of 4 pencils he would make a total profit of Re 1, whereas by investing it in the purchase of 1 note-book his profit would only be Re 0.50. Some possible values of x and y with the resulting profit are given in the able below:

x	y	*cost(Rs.)*	*profit (Rs.)*
1	33	99.75	16.75
5	32	99.75	17.25
53	20	99.75	23.25
21	24	87.75	17.25
33	20	84.75	18.25
133	—	99.75	33.25

We see that his profit is maximum if he invests all his capital in the purchase of pencils only. Then he utilises almost all his capital and makes a profit of Rs 33.25.

Suppose the shopkeeper knows from past experience that the sales of note-books cannot exceed 25, and therefore he should not purchase more than 25 note-books. Suppose he also knows that customers of note-books usually also buy pencils, and that if pencils run out of stock it becomes difficult to sell note-books. Accordingly, he decides that to begin with the number of pencils in stock should be at least four times the number of note-books. Thus, his choice of x and y is now determined by the following conditions:

$$0.75\,x + 3y \leq 100$$
$$y \leq 25$$
$$x \geq 4y$$

Of course, x and y should be positive integers.

The possible values of x and y which satisfy these conditions will be such tha the point (x, y) lies in the shaded region in the figure below:

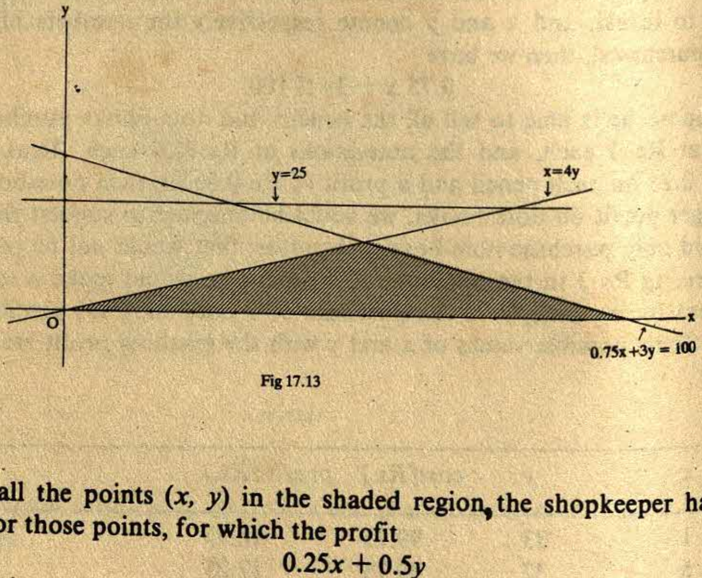


Fig 17.13

Out of all the points (x, y) in the shaded region, the shopkeeper has to choose that point, or those points, for which the profit

$$0.25x + 0.5y$$

has the maximum value. Some possible values of x and y with the resulting profit are given in the table below:

x	y	*cost (Rs.)*	*profit (Rs.)*
80	13	99	26.5
84	11	96	26.5
116	4	99	31
129	1	99.75	32.75

We notice that the shopkeeper gets the same profit (Rs 26.5) if he invests Rs 99 or Rs 96. He would no doubt prefer to invest Rs 96 instead of Rs 99 if he knows that in both cases his profit is the same. In view of this the shopkeeper can look at the problem in a different manner. He can fix a minimum amount of profit to be made and they try to achieve it with as small an investment as possible.

Suppose the shopkeeper decides that his profit should not be below Rs 30. He has now to choose x and y in such a manner that he can make a profit of Rs 30 or more keeping his investment as low as possible. Now, the variables x and y have to satisfy only one condition, namely

$$0.25x + 0.5y \geq 30,$$

in addition to being non-negative integers. Out of all such pairs of values (x, y), he chooses that for which

$$0.75x + 3y$$

has the smallest value.

The two situations described above illustrate the type of problems called linear rogramming problems. The term *linear programming* refers to the special methods volved for solving such problems. We give below a brief description of these ethods.

athematical Model of Linear Programming Problems

e have two variables x_1 and x_2 which are non-negative (i.e., $x_1 \geq 0$, $x_2 \geq 0$). These ariables have also to satisfy a number of *linear* constraints. Since a linear constraint $x_1 + 4x_2 \geq 5$ can also be expressed as $-3x_1 - 4x_2 \leq -5$, we write all the constraints f the problem in the form

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 \leq b_2$$

nd so on. We are also given a *linear* function

$$c_1x_1 + c_2x_2$$

hose value is to be maximised (or minimised) subject to the given constraints. In her words, out of all pairs (x_1, x_2) satisfying the constraints we have to choose ose which give a maximum (or minimum) value to the given linear function. A ear programming problem when expressed in mathematical form looks as follows:
ind max $(3x_1 - 2x_2)$ given that

$$x_1 + x_2 \leq 1.$$
$$3x_1 - x_2 \leq -3$$
$$x_1, x_2 \geq 0$$

r it may be;
ind min $(1.5x_1 + 2.5x_2)$ given that

$$-x_1 - 3x_2 \leq -3$$
$$x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

The set of pairs of values $(x_1,$ [illegible]$)$ which satisfy the linear constraints of the roblem is called the set of *feasible solutions* to the problem. The linear function hose value is to be maximised (or minimised) is called the *objective function*.

17.3 Solution of a Linear Programming Problem

In most linear programming problems the set of feasible solutions is a *polygon* in the positive quadrant, that is, a closed figure bounded by straight lines. The set is also a *convex set* which only means that if you take any two points in the set the line joining them also lies in the set. The set of feasible solutions is thus a *convex polygon*. In figure 17.4 below polygon at (i) and (ii) are convex, while that at (iii) is not.

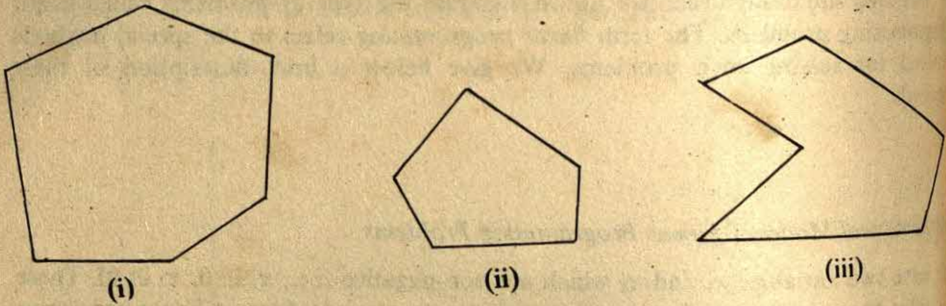

(i) (ii) (iii)

Fig 17.14

Once we have determined this convex polygon, we have to select a point, or points in it which will make the value of the objective function maximum (or minimum). Since, the objective function is linear, we do not have to consider all the points inside the polygon to locate the point which maximises (or minimises) the objective function. It is known from general mathematical theory that it is enough to look at the values of the objective function at the vertices of the set of feasible solutions. The largest of these values is the maximum value of the objective function, and the smallest of these values is the minimum.

It may happen that two vertices give the same value of the objective function which also happens to be the largest value. In that case all the points on the line joining these two vertices will give the maximum value of the objective function. Similar is the case when two vertices give the same value which is the smallest of all the values. Then all the points on the line joining these two vertices will give the minimum value to the objective function.

The different methods of solving linear programming problems make use of the above property of linear functions. One begins by obtaining the value of the objective function at one of the vertices and then moving to another vertex which increases the value of the objective function. The procedure is continued till one reaches the maximum value. The procedure for obtaining the minimum value is similar.

We now describe a geometrical method to solve a linear programming problem. This method locates the vertex of maximum (or minimum) value without having to evaluate the value of the objective function at any other vertex. Consider the following linear programming problem.

Maximise $x + 2y$ under the constraints

$$\begin{aligned} 2x + 3y &\leq 6, \\ x + 4y &\leq 4, \\ x, y &\geq 0. \end{aligned}$$

The set of feasible solutions is represented by the shaded area *OPQR* in the figure below: (See exercise 17.1 problem no. 1)

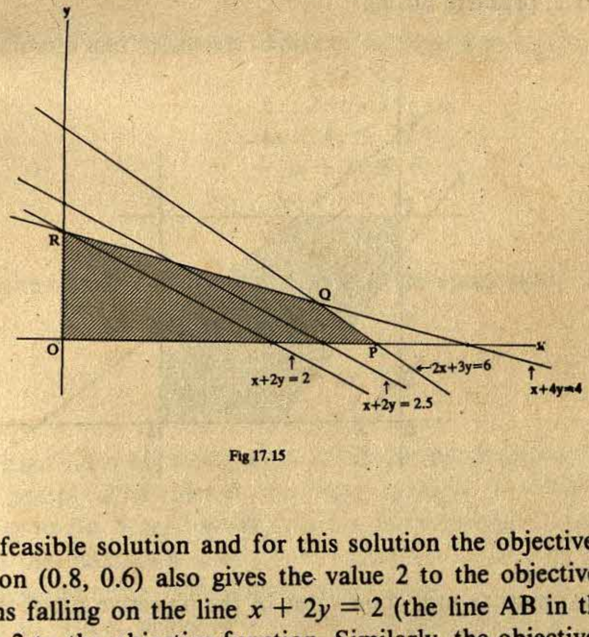


Fig 17.15

$x = 1$, $y = .5$ is a feasible solution and for this solution the objective function has value 2. The solution (0.8, 0.6) also gives the value 2 to the objective function. In fact all the solutions falling on the line $x + 2y = 2$ (the line AB in the figure) will give the same value 2 to the objective function. Similarly, the objective function has value 2.5 for all solutions (x, y) for which $x + 2y = 2.5$. These solutions fall on the line *CD* which is parallel to *AB* and away from the origin. Thus, as we move the line *AB* parallel to itself and away from the origin, we get feasible solutions giving larger and larger values to the objective function. The value of the objective function keeps increasing till we get a line parallel to *AB* and passing through the vertex *Q* of the polygon of feasible solutions. Any further displacement of the line takes it out of the polygon *OPQR* so that the value of the objective function corresponding to the line is no longer attained by any feasible solution. The maximum value of the objective function takes place at the vertex *Q* and equals $\frac{12}{5} + 2 \times \frac{2}{5} = \frac{16}{5} = 3.2$. If we move the line *AB* parallel to itself but towards the origin, the value of the objective functions keeps decreasing. The minimum value is reached at the vertex O and equals zero.

As another example consider the set of feasible solutions of the constraints.

$$\begin{aligned} -x-y &\leq -1 \\ 7x+9y &\leq 63 \\ x &\leq 6 \\ y &\leq 5 \\ x, y &\geq 0. \end{aligned}$$

It is given by the polygon $ABCDEF$ in the figure below; (See exercise 17.1, problem no. 2).

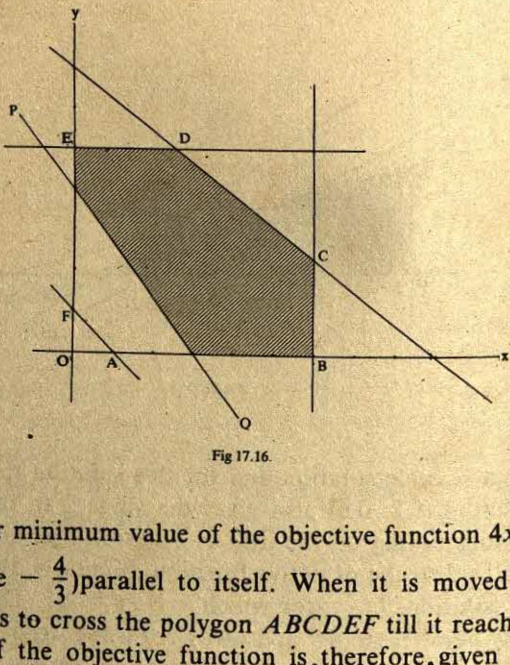


Fig 17.16.

To find the maximum or minimum value of the objective function $4x + 3y$, we move the line PQ (with slope $-\frac{4}{3}$)parallel to itself. When it is moved away from the origin, the line continues to cross the polygon $ABCDEF$ till it reaches the vertex C. The maximum value of the objective function is, therefore, given by the solution represented by the vertex C. Similarly, by moving the line towards the origin, we see that the minimum value of the objective function is determined by the vertex F.

The coordinates of the six vertices of the polygon $ABCDEF$ are given by

$$A: (1, 0),\ B: (6, 0),\ C: (6, \tfrac{7}{3})$$

$$D: (\tfrac{18}{7}, 5),\ E: (0, 5.),\ F: (0, 1)$$

The values of the objective function $4x + 3y$ at these vertices are as follows;

$$A: 4,\ B: 24;\ C: 31,$$

$$D: 25\tfrac{2}{7},\ E: 15;\ F: 3$$

and we verify that among the vertices the maximum value of the objective function is at C and minimum is at F.

EXERCISE 17.2

1. Find the maximum and minimum values of $5x + 2y$ subject to the constraints

$$\begin{aligned} -2x - 3y &\leq -6 \ (i.\,e.,\ 2x + 3y \geq 6) \\ x - 2y &\leq 2 \\ 6x + 4y &\leq 24 \\ -3x + 2y &\leq 3 \\ x, y &\geq 0 \end{aligned}$$

2. Find the maximum and minimum values of $2x+y$ subject tothe constraints

$$\begin{aligned} x + 3y &\geq 6 \\ x - 3y &\leq 3 \\ 3x + 4y &\leq 24 \\ -3x + 2y &\leq 6 \\ 5x + y &\geq 5 \\ x, y &\geq 0. \end{aligned}$$

3. Find the minimum value of $3x + 5y$ subject to the constraints

$$\begin{aligned} -2x + y &\leq 4 \\ x + y &\geq 3, \\ x - 2y &\leq 2 \\ x, y &\geq 0 \end{aligned}$$

4. If a young man rides his motor-cycle at 25 km per hour, he has to spend Rs 2 per km on petrol; if he rides it at a faster speed of 40 km per hour, the petrol cost increases to Rs 5 per km. He has Rs. 100 to spend on petrol and wishes to find what is the maximum distance he can travel within one hour. Express this as a linear programming problem and then solve it.
5. Every gram of wheat provides 0.1 g of proteins and 0.25 g of carbohydrates. The corresponding values for rice are 0.05 g and 0.5 g respectively. Wheat costs Rs. 2 per kg and rice Rs. 8. The minimum daily requirements of protein and carbohydrates for an average child are 50 g and 200 g respectively. In what quantities should wheat and rice be mixed in the daily diet to provide the minimum daily requirements of protein and carbohydrates at minimum cost. (The protein and carbohydrate values given here are fictitious and may be quite different from the actual values.)

CHAPTER 18

ALGORITHMS AND FLOWCHARTS

18.1 Computers

The advent of computer is affecting practically every activity of human life. Whether it is reservation in railways, managing a super-market, or clearing cheques in banks, we have begun using computer as a tool everywhere. Now-a-days in India, every big organisation is using computer for its day-to-day working. We use computer to locate underground natural gas and oil, to launch a satellite, and so on. In fact, the invention of computer is one of the finest contributions of science to society.

The development of the ideas leading to the present day computer dates back to the nineteenth century. Some Mathematicians like Charles Babbage (1791-1871), Alan Turing (1912-1954) and John Von Neumann (1903-1957) contributed significantly to the concept of a computing machine. Mathematics and Computer enrich each other. The present-day computer has become a powerful tool to solve some mathematical problems that remained unsolved hitherto.

A computer, however, is not a machine that can do more mathematical operations than what a human being can do. But it can do some operations much faster and can handle large amount of data. You may be surprised to know that the present-day computer can perform a million additions per second. It can perform all operations such as addition, subtraction, multiplication, division and comparison.

There has been a variety of computers, depending on their purpose and capabilities. One common feature is that every computer is constituted of three major components: 1) Input and output component, 2) Central processing unit, and 3) Memory.

Depending on the nature of these three components, the computers vary. All the computers do the same functions with varying degrees of speed and limits.

However, when a problem is given to the computer, it cannot automatically decide the operations and the operands. It is necessary for a person, desirous to use a computer for solving a problem, to instruct the computer properly to choose the operations and the operands. This requires a mode of communication with a computer. Imagine that Chinese wants to tell a story to an Englishman. Though the Chinese knows the story, unless he knows a language that the Englishman understands, he cannot actually communicate with him. Similarly, if any instruction is to be given to the compurter, it is necessary that instructions are delivered in a language that the computer understands. Such languages are called Programming Languages: FORTRAN, BASIC, PASCAL, COBOL, etc. Irrespective of language, what is equally important, is to prepare the sequence of instructions to solve the specific problem. That is what we call an *algorithm*.

18.2 **Algorithms**

An algorithm is nothing but a sequence of simple steps to solve the problem at hand. It is a step-by-step procedure leading to the solution.

For example, to solve the equation $ax + b = c$,

we may write: Step 1 : Subtract b from c

Step 2 : Divide this answer by a, if $a \neq 0$

Step 3 : This answer as the value of x is the solution.

Some characteristic features of an algorithm are:

(i) It is precise. There can be no ambiguity in a computer algorithm. Each step of the execution must be uniquely defined and may depend only on the inputs, the previous steps and the internal capabilities of the machine.

(ii) It is sufficiently detailed and precise to allow execution by the processor.For example, "find the area of this triangle", "solve the quadratic equation", etc. are instructions that require more details. In an algorithm, we have to incorporate how the area is to be found, or how the roots of the quadratic equation are to be determined. There is no room left for the creative imagination of the executor. But while planning an algorithm, some details may be suppressed.

(iii) It is in a definite order. The instructions in an algorithm are to be given in the order in which they are to be performed. The machine carries out these instructions, one after the other.

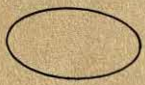
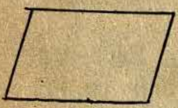
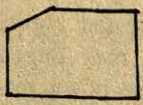
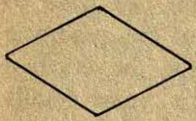
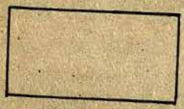
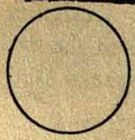
We say an algorithm, which is designed for a specific problem, is correct if it serves for all instances of the problem. Designing and studying computer algorithm occupies a unique and central position in the study of computer. Designing of algorithms is nothing but planning. Though algorithms occur in some of the earliest of human records, it is the rise of computers that has led to the widespread study of algorithm for their own sake.

18.2 **Flow Charts**

Before beginning to solve a problem, it is frequently useful to do some preliminary planning. A graphic representation of such a plan, is called a *flow-chart*. The flow charts constitute schematic pictures that we intend to implement in the program. The flow chart helps to break a big job down into many small pieces of the job and to represent them pictorially, showing the order of instructions.

A flow chart consists of some boxes, linked by some arrows. In each box, certain action to be carried out, is mentioned. Arrows on the lines connecting the boxes indicate the direction in which we should proceed.

The boxes are of different shapes as shown below:

Shape of the box	*Meaning*
 a stretched ellipse	Terminal box: To begin or end a program.
 or  a parallelogram	Input/output box: The data fed into the computer and the print out given by the computer
 a rhombus or a diamond	Decision box: Computer is to decide among alternative instructions
 a rectangle	Assignment/calculation box: Computer is to assign some values for the variables, or is to perform some operations like addition, etc.
 a circle	Connector box: To come from or to go to another part of the chart

We start with an easy example.

Example 18.1

Problem: Find the modulus of a complex number given in the form (X, Y)
Method: The modulus of (X, Y) is given by the formula $r = \sqrt{X^2 + Y^2}$

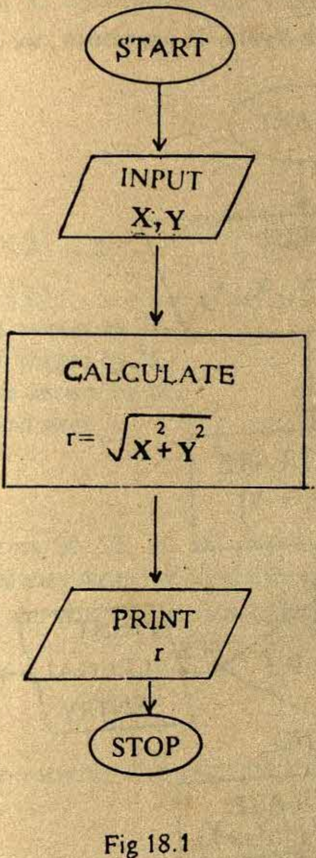


Fig 18.1

Next, let us see a less easy problem.

Example 18.2

Problem: Given two complex mumbers $Z_1 = (X_1, Y_1)$

and $Z_2 = (X_2, Y_2)$; compute $\frac{Z_1}{Z_2}$.

Method: If $\frac{Z_1}{Z_2}$ is (X_3, Y_3), then

$$X_3 = \frac{X_1 X_2 + Y_1 Y_2}{X_2^2 + Y_2^2}$$

and $$Y_3 = \frac{Y_1 X_2 - Y_2 X_1}{X_2^2 + Y_2^2}$$

When $Z_2 = 0$, that is when both X_2 and Y_2 are zeroes , we do not calculate $\frac{Z_1}{Z_2}$.

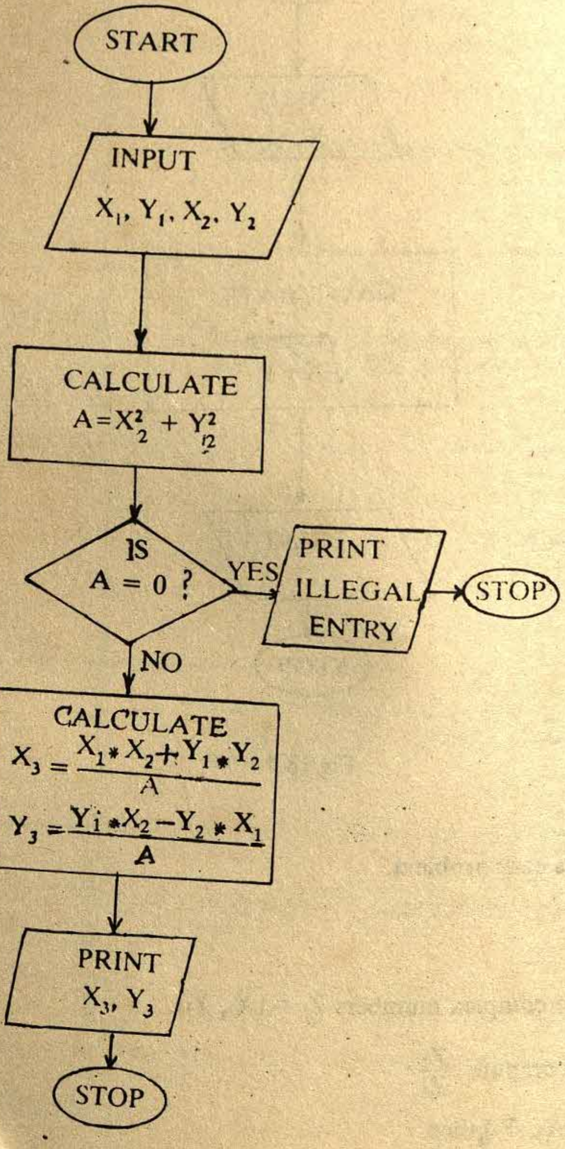


Fig 18.2

18.4 Loops

In this class you will be taught how to assign different values for a variable, and create a loop while writing an algorithm. This is best explained by means of an example.

Example 18.3:

Multiply $20 \times 22 \times 24 \times 26 \times 28$

Method: First multiply 20×22.
Then multiply this answer by 24.
Then multiply this answer by 26.
Lastly multiply this answer by 28.
Print the answer and stop.

Now, we note that the factors 20, 22, 24, 26, and 28 are progressively having an increment of 2. Therefore, we may write the algorithm as follows:

Note: In computer language the symbol * denotes multiplication.

```
$X = 20$
$A = 20$
$X \leftarrow X + 2$ [This means, the variable $X$ is assigned a new value, equal to (old value plus 2)]
        [Now $X = 22$]
$A \leftarrow A * X$ [Now $A = 20 \times 22$]
$X \leftarrow X + 2$ [Now $X = 22 + 2 = 24$]
$A \leftarrow A * X$ [Now $A = (20 \times 22) \times 24$]
$X \leftarrow X + 2$ [Now $X = 24 + 2 = 26$]
$A \leftarrow A * X$ [Now $A = (20 \times 22 \times 24) \times 26$]
$X \leftarrow X + 2$ [Now $X = 26 + 2 = 28$]
$A \leftarrow A * X$ [Now $A$ is the required answer]
Print $A$ and stop.
```

In this algorithm, we find that the two lines $X \leftarrow X + 2$ and $A \leftarrow A * X$ are repeated many times. There is way to write this briefly, as shown in the flow chart below:

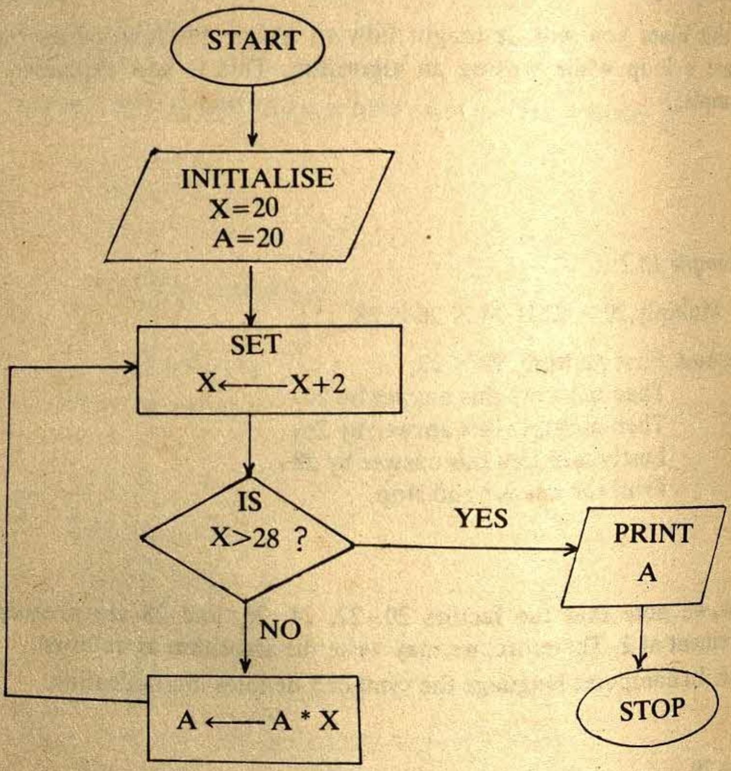
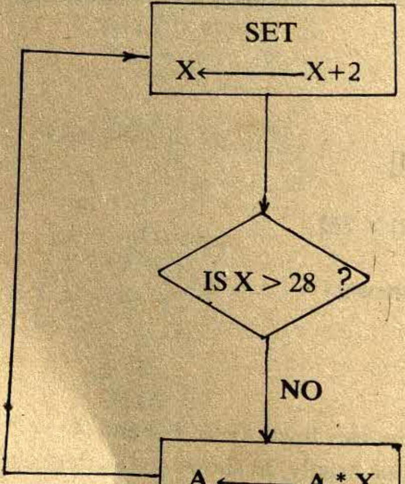
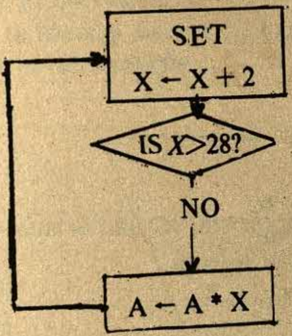


Fig 18.3

The portion



Explanation: The portion



is a loop. It is traced again and again. When do we go out of the loop 7. Only when the answer to the decision box 'Is $X>28$?' is '*YES*'. At that time the value of A becomes $20 \times 22 \times 24 \times 26 \times 28$.

18.5 Summary

In Example 18.1, we saw an algorithm of the easiest type where the problem involved calculations only.

In Example 18.2, we saw algorithm for a problem involving both calculations and decision-making.

In Example 18.3, we saw an algorithm that involves calculations, decisions and also loops.

In the examples that follow, we take some problems from the earlier chapters of this book and write down alogorithms for solving them. The Examples 18.4 to 18.6 will be accompanied by actual working of the algorithm in a particular instance. In the later example, only the flow chart and its method will be given.

Example 18.4

Problem: To find the intersection of a finte number of finite sets.
*Method:*Let the given sets be $A_1, A_2, .., A_n$.

Arrange them in such a way that A_1 has the least number of elements, when compared with $A_2,...,A_n$ (this facilitates a quicker solution).

Let $A_1 = \{x_1, x_2, ..., x_m\}$
Is $x_1 \in A_2$?
If not, leave x_1 and go to x_2.
Else, is $x_1 \in A_3$?
and so on.
If yes for all questions up to "Is $x_1 \in A_n$?", print x_1. Otherwise,don't print x_1.
Then do the same for x_2, x_3 and so on.
The printed elements are the elements of the intersection.

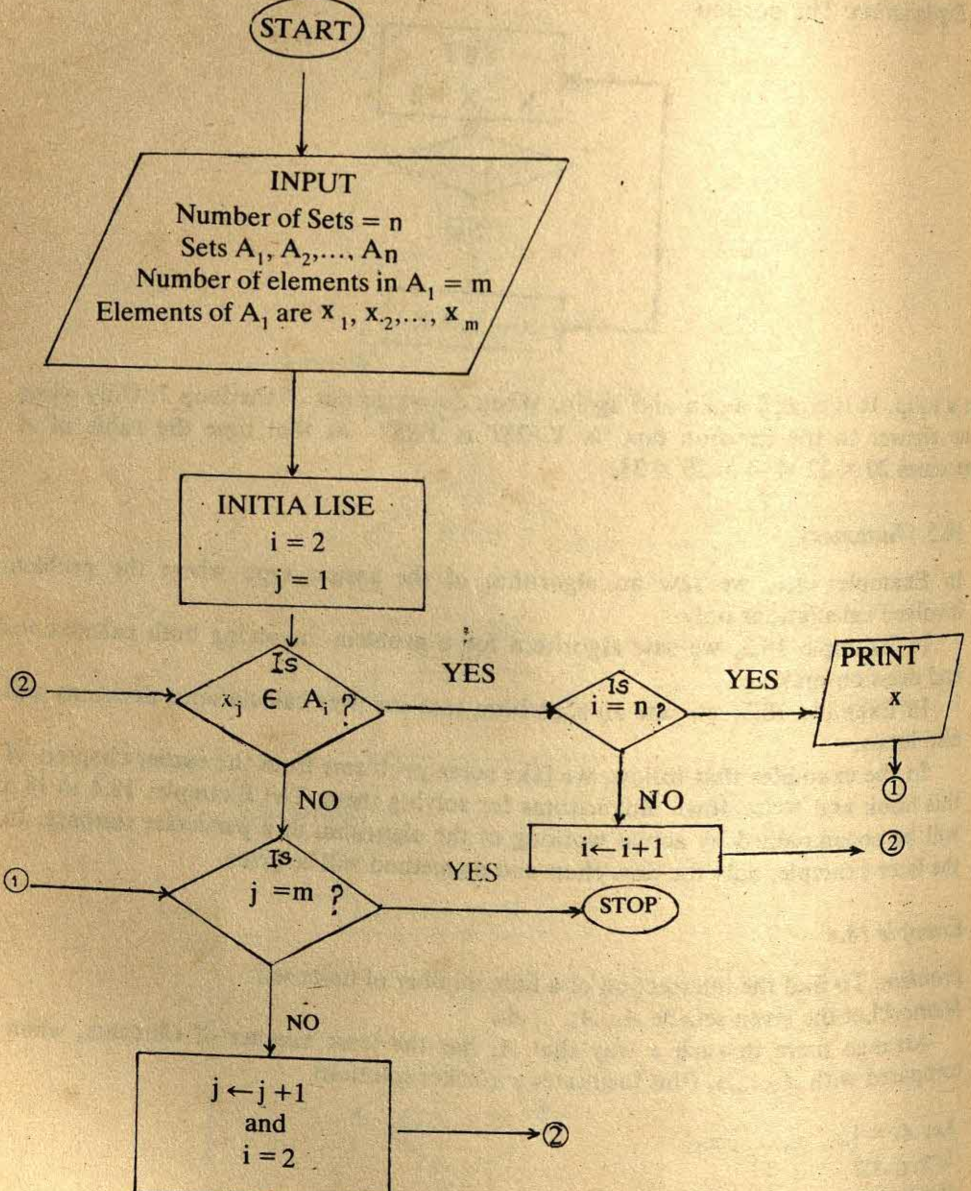

START
INPUT
Number of Sets = n
Sets A1, A2,..., An
Number of elements in A1 = m
Elements of A1 are x1, x2,..., xm
INITIA LISE
i = 2
j = 1
Is xj ∈ Ai ?
YES
Is i = n ?
YES
PRINT x
①
NO
i← i+1
②
②
NO
Is j = m ?
①
YES
STOP
NO
j ← j + 1
and
i = 2
②

Fig 18.4

Remark:

After $\boxed{\text{Print } x_j}$ we have the arrow towards ①
This means, we should go to the step marked ① namely, the decision box $\boxed{j = m}$
Let us see a particular case of this example.

Let $A = \{1, 2, 3, 4, 5\}$
$B = \{3, 4, 5, 8, 9\}$
$C = \{2, 3, 8, 9\}$
$D = \{1, 2, 3, 10\}$
be four sets, for which we want to find the intersection. We rename them as
$A_1 = C = \{2, 3, 8, 9\}$
$A_2 = A = \{1, 2, 3, 4, 5\}$
$A_3 = B = \{3, 4, 5, 8, 9\}$
and $A_4 = D = \{1, 2, 3, 10\}$
Here $m = 4$, $n = 4$, $x_1 = 2$, $x_2 = 3$, $x_3 = 8$, $x_4 = 9$.
Is $x_1 \in A_2$?
Yes, because $2 \in A_2$
Is $1 = n$? No, because $n = 4$.
New $i = 3$.
Is $x_1 \in A_3$?
No, because $2 \notin A_3$.
Is $3 = m$? No, because $m = 4$
New $j = 2$ and $i = 2$
Is $x_2 \in A_2$? Yes, because $3 \in A_2$.
Is $2 = n$? No, because $n = 4$.
New $i = 3$.
Is $x_2 \in A_3$? Yes, $3 \in A_3$.
Is $i = n$? No, $3 \neq 4$.
New $i = 4$.
Is $x_2 \in A_4$? Yes, $3 \in A_4$.
Is $i = n$? Yes, $i = n = 4$.
Print x_2. Now 3 is printed
Is $j = m$? No, $2 \neq 4$.
New $j = 3$ and $i = 2$.
Is x_3 in A_2? No, $8 \notin A_2$.
Is $j = m$? No, $3 \neq 4$.
New $j = 4$ and $i = 2$.
Is x_4 in A_2? No, $9 \notin A_2$?
Is $j = m$? Yes, both $= 4$.
Stop.
After stopping, what are all the things already printed out? 3 only.
Then we conclude that 3 is the only element of the intersection.
That is, $A \cap B \cap C \cap D = \{3\}$.

Remark

When m and n are large i.e., when too many big sets are given, using this algorithm in a computer will be faster than finding the intersection without the computer.

Example 18.5

Problem: Check whether a given finite sequence of numbers is in geometric progression.

Method: Let the given sequence be

$a_1, a_2, \ldots, a_n$

This sequence is in G.P. if and only if

$a_i^2 = a_{i-1}\, a_{i+1}$ for $2 \leq i \leq n - 1$.

Step 1: Are there at least three terms? If not, conclude that it is in G.P. and stop.

Step 2: Else, verify if $a_2^2 = a_1 a_3$.
If not, conclude that it is not in G.P. and stop.

Step 3: Else, verify if $a_3^2 = a_2 a_4$.
If not, conclude that it is not in G.P. and stop. Else, proceed.

Step 4: Go on, until $a^2_{n-1} = a_{n-2} a_n$ is verified.
If yes until that, conclude that it is in G.P. and stop.

Particular Instance: Is 2, 6, 18, 36 in Geometric Progression?

Here, $n = 4$

$a_1 = 2$
$a_2 = 6$
$a_3 = 18$
$a_4 = 36$

Are there at least 3 terms? i.e., is $n > 3$?
Yes.

Initially $i = 2$
$A = 2$
$B = 6$
$C = 18$

Is $B^2 = AC$
$6^2 = 18 \times 2$?
Yes.

New $i = 3$
Is $i = n$?
No.

Set $A = 6$
$B = 18$
$C = 36$

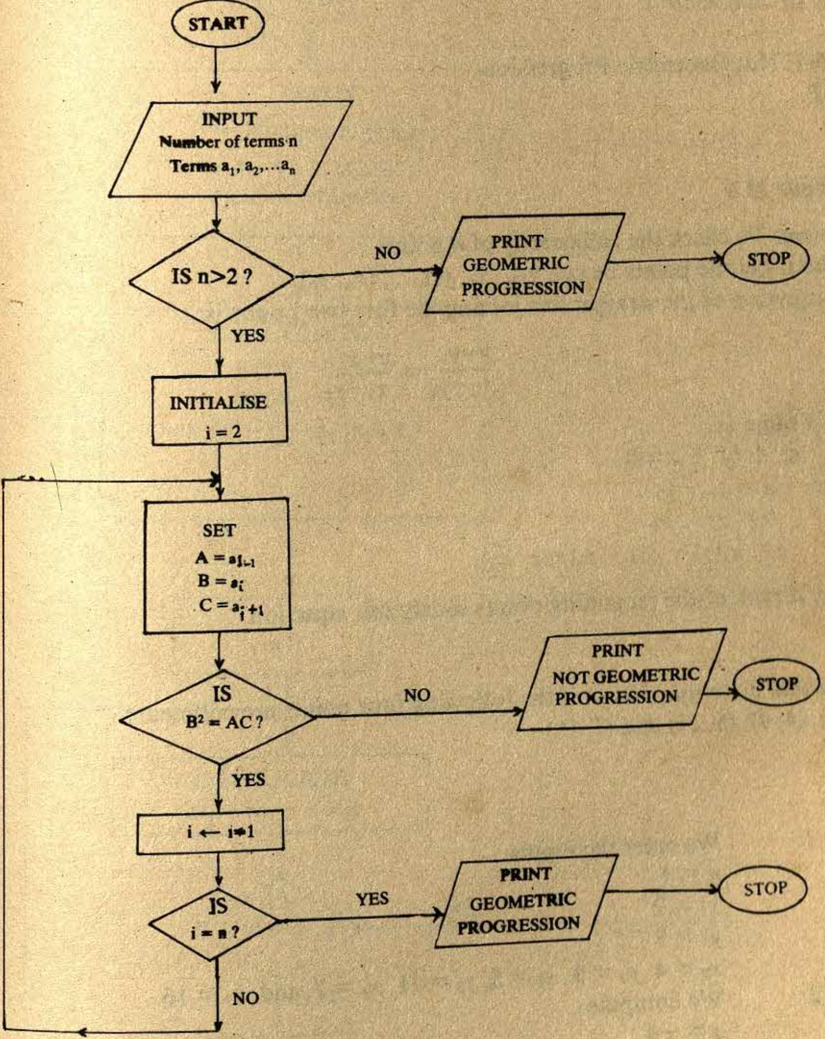

START
INPUT
Number of terms n
Terms a1, a2,...an
IS n>2 ?
NO
PRINT
GEOMETRIC
PROGRESSION
STOP
YES
INITIALISE
i = 2
SET
A = ai-1
B = ai
C = ai+1
IS
B² = AC ?
NO
PRINT
NOT GEOMETRIC
PROGRESSION
STOP
YES
i ← i+1
IS
i = n ?
YES
PRINT
GEOMETRIC
PROGRESSION
STOP
NO

Fig18.5

Is $B^2 = AC$?
i.e. Is $18^2 = 6 \times 36$?
No.
PRINT: Not Geometric Progression.
STOP.

Example 18.6

Problem: To check the collinearity of n points.
Method: Let the points be $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
The equation of the straight line joining the first two points is:

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

This is same as
$ax + by + c = 0$
where $a = y_1 - y_2$
$b = x_2 - x_1$
and $c = x_1(y_2 - y_1) - y_1(x_2 - x_1)$

Verify if each of the remaining points satisfy this equation.

Particular Example: Verify if the following four points are collinear:
(2, 5), (4, 9), (5, 11) and (7, 16)

Step 1: We enter the inputs
$n = 4$
$x_1 = 2$
$y_1 = 5$
$x_2 = 4, y_2 = 9, x_3 = 5, y_3 = 11, x_4 = 7$ and $y_4 = 16$

Step 2: We compute
$a = -4$
$b = 2$
$c = -2$

Step 3: $ax_3 + by_3 + c = 0$ is true

Step 4: $i = 4$
Is $4 > n$? No.

Step 5: Compute $d = ax_4 + by_4 + c = 2 \neq 0$; hence
output: the points are not collinear.

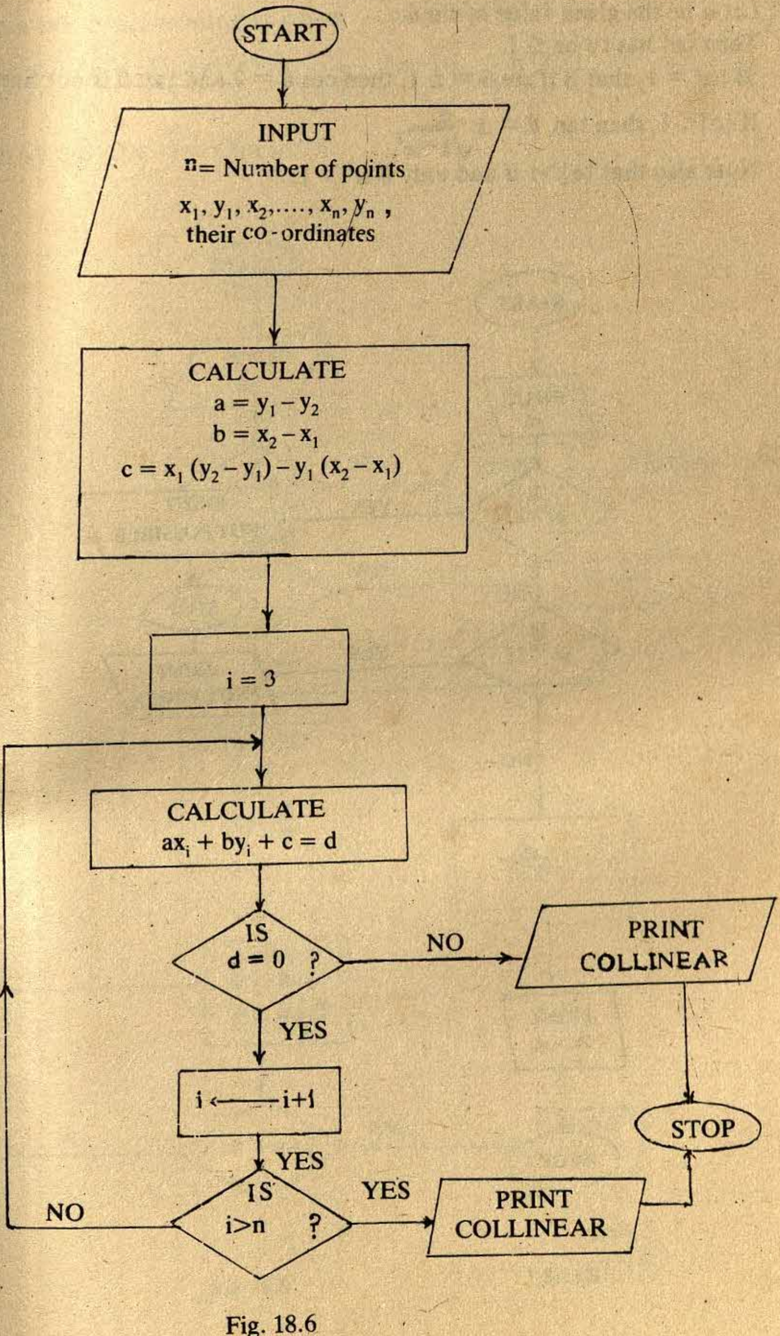

START
INPUT
n= Number of points
$x_1, y_1, x_2, \ldots, x_n, y_n$,
their co-ordinates
CALCULATE
$a = y_1 - y_2$
$b = x_2 - x_1$
$c = x_1 (y_2 - y_1) - y_1 (x_2 - x_1)$
$i = 3$
CALCULATE
$ax_i + by_i + c = d$
IS
$d = 0$?
NO
PRINT
COLLINEAR
YES
$i \longleftarrow i+1$
YES
IS
$i>n$?
YES
NO
PRINT
COLLINEAR
STOP

Fig. 18.6

Example 18.7

Problem: Given the value of sin θ, find the value of tan θ.

Method: Let α be the given value of sin θ

Then $|\alpha|$ has to be ≤ 1

If $|\alpha| = 1$, that is if sin $\theta = \pm 1$, then cos $\theta = 0$ and tan θ is not finite.

If $|\alpha| < 1$, then tan $\theta = \pm \frac{\alpha}{\sqrt{1-\alpha^2}}$

Note also that $|\alpha| > 1$ if and only if $\alpha^2 > 1$.

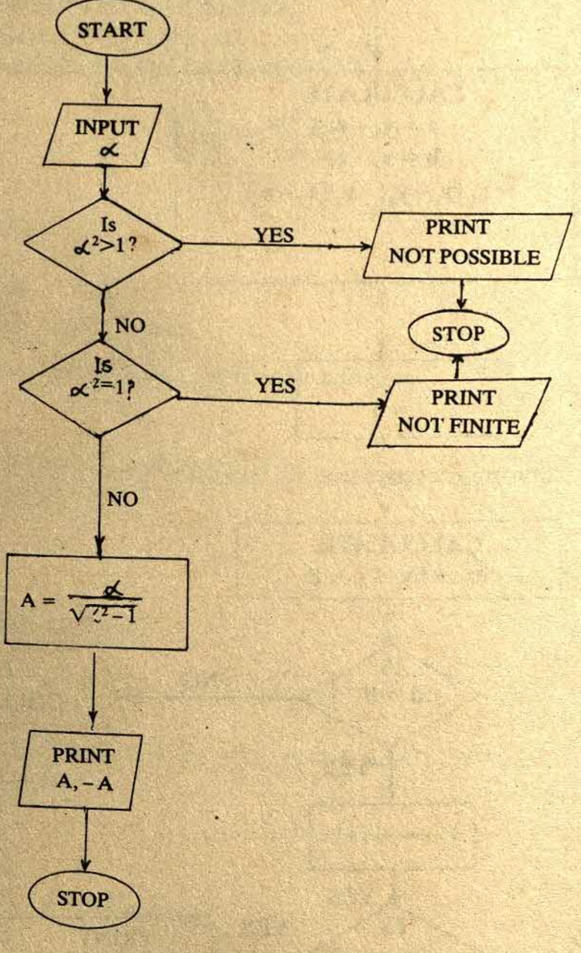


Fig. 18.7

Example 18.8

Problem: Given two positive integers n and r, compute $C(n, r)$.

Method: $C(n, r) = \dfrac{n(n-1)\ldots(n-r+1)}{1.2.\ldots r} = \left(\frac{n}{1}\right)\left(\frac{n-1}{2}\right)\ldots\left(\frac{n-r+1}{r}\right)$

If $r > \frac{n}{2}$, then we use $C(n, r) = C(n, n-r)$.

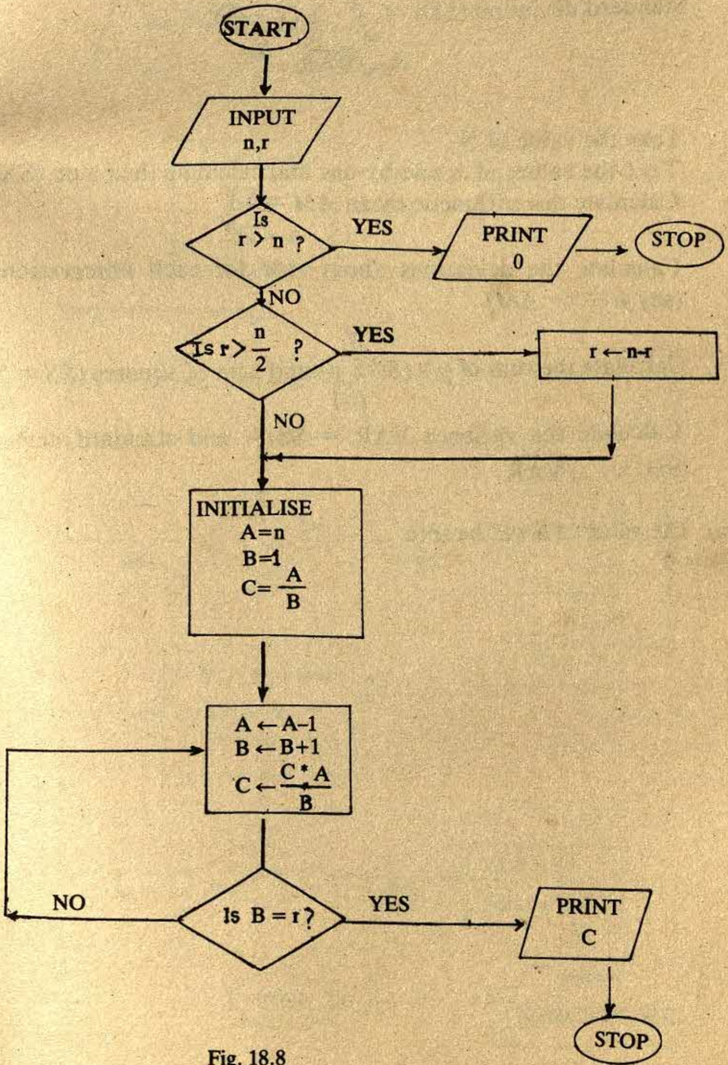


Fig. 18.8

Example 18.9

Problem: Calculate the variance and standard deviation for the given raw data.

Method: Variance (VAR) $= \frac{1}{N} \Sigma (x_i - \bar{x})^2$

N is no of observations

x_i is i th observation,

$\bar{x}$ is arithmetic mean (AM)

Standard deviation (SD) $= \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{N,}}$

$= \sqrt{\text{VAR}}$

Step 1: Take the value of N

Step 2: Take the values of x_i one-by-one and calculate their sum (SX).

Step 3: Calculate the arithmetic mean $AM = \frac{SX}{N}$

Step 4: Calculate the deviations about AM for each observation (say $y_i = x_i - AM$)

Step 5: Calculate the sum of y_i's $(S = \sum_{i=l}^{N} y_i)$ and sum of squares $(SS = \sum_{i=l}^{N} y^2_i)$.

Step 6: Calculate the variance VAR = SS/N and standard deviation, SD = $\sqrt{\text{VAR}}$

Observation: The value of S will be zero

$\therefore \Sigma (x_i - \bar{x}) = 0$

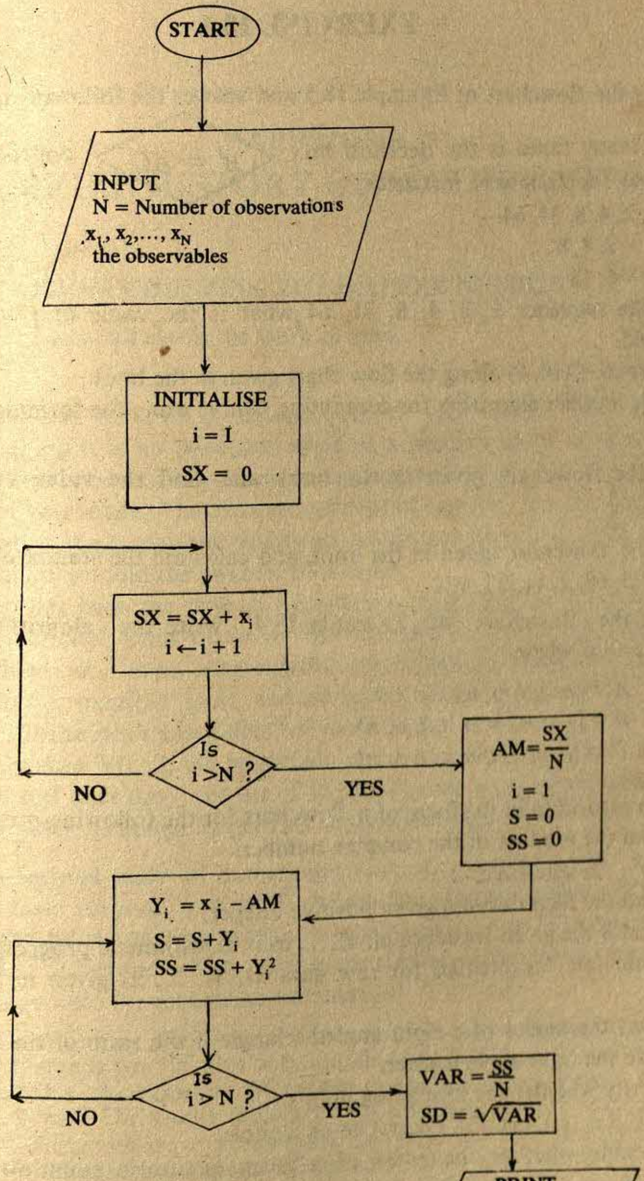

START
INPUT
N = Number of observations
$x_1, x_2, \dots, x_N$
the observables
INITIALISE
$i = 1$
$SX = 0$
$SX = SX + x_i$
$i \leftarrow i + 1$
Is $i > N$?
NO
YES
$AM = \frac{SX}{N}$
$i = 1$
$S = 0$
$SS = 0$
$Y_i = x_i - AM$
$S = S + Y_i$
$SS = SS + Y_i^2$
Is $i > N$?
NO
YES
$VAR = \frac{SS}{N}$
$SD = \sqrt{VAR}$
PRINT
N, AM, VAR, SD, S
STOP

Fig. 18. 9)

EXERCISE 18.1

1. Consider the flowchart of Example 18.5 and answer the following questions:
   - (a) How many times is the decision box $B^2 = AC$ approached in each of the following instances:
     - i) 1, 2, 4, 8, 31, 64
     - ii) 1, −2, 4, 8
     - iii) 2, −6, 18
   - (b) For the sequence 1, 2, 4, 8, 31, 64, what is the value of i when STOP is reached.
2. (a) Compute $C(10, 6)$ along the flow chart given in the book.
   - (b) Write another algorithm for computing $C(n, r)$ using the formula $\frac{n!}{r!(n-r)!}$
3. (a) Use the flowchart given in the book and find the value of tan θ when $\sin\theta = \frac{1}{2}$.
   - (b) Use the flowchart given in the book and calculate the standard deviation of the data: 25, 60, 2, 11, 23, 6, 1
   - (c) Using the flowchart of Example 18.4, write the alogrithm for finding $A \cap B \cap C$ where

$$A = \{a, e, i, o, u\}$$
$$B = \{e, d, u, c, a, t, i, o, n\}$$
and $$C = \{e, v, o, l, u, t, i, o, n\}$$

4. Write an algorithm in the form of a flowchart for the following problems:
   - (a) To find the product of the complex numbers $z_1, z_2, ..., z_n$ where $n \geq 2$.
   - (b) To find the factorial of a given positive integer.
   - (c) To find if the given sequence $a_1, a_2, ..., a_n$ is in arithmetic progression.
   - (d) To calculate the median for raw data $x_1, x_2, ..., x_n$ given in the increasing order.
   - (e) To find the angles of a right-angled triangle if the ratio of the two sides that include the right angle is given.
   - (f) To verify whether the four given points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ lie on a circle.
   - (g) To decide whether the roots of a given quadratic equation are real and distinct.
   - (h) To calculate the arithmetic mean of the given n numbers.

*APPENDIX
A BRIEF HISTORY OF MATHEMATICS

FROM COUNTING TO COMPLEX NUMBERS

God made integers, all else is the work of man.

L. K.RONECKER

In our times the method of counting looks very simple. But actually and historically speaking it is an advanced stage of a process which is by no means as innate as it appears. Several stages were crossed before the epochal discovery of cardinal number was made. The abstract concept of number was preceded by ability for counting which itself required ability to match the objects of a collection with those of an auxiliary set and the awareness of order.

In India, ten has been the base of numeration since very early times. Names of decimal denominational terms occur in the *Vajasaneyi Samhita* (of the *Yajurveda)* XVII, 2, as follows: *eka, dasa, sata, sahasra, ayuta, niyuta, prayuta* ($=10^6$), *arbuda, nyarbuda, samudra, madhya, anta,* and *parardha* which stands for 10^{12}. Later on this list, called Medhātithi's List (after the name of the Vedic seer), was extended in the *Taittiriya Saṁhita* VII, 2, 20, to include seven more terms, the last being called *loka* ($=10^{19}$). In still later days, the list of 18 terms given by Sridhara (about 750 AD) became more popular, although longer lists continued to appear.

Several non-decimal scales of numeration were also prevalent. The first count is the centesimal scale (beyond *koti* $=10^7$) which the young Gautama Buddha gave (when interviewed by the mathematician Arjuna) consisted of 23 terms ending with *tallaksana* ($= 10^{7+46}$) beyond which eight more similar counts were there. Thus the whole series will go upto the monstrous number $10^{53+8\times46} = 10^{421}$

The highest terminolgy the Greeks had was their word *myriad* (form *myrioi*, 'countless') which stands for 10^4. But without adequate terminology and notation the genius of Archimedes (third century BC) could devise very large periods of counting in his *Sand Reckoner*. The words million ($= 10^6$), billion, trillion, etc. were coined quite late (about thirteenth certury onwards), but were not fully adopted in English until the seventeenth century. Recently Edward Kasner has introduced the names "googol" which stands for 10^{100}, and "googolplex" which denotes ten to the power googol.

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*This appendix is condensed from the material contributed by Prof. R.C. Gupta, Professor of Mathematics, Birla Institute of Technology, Mesra, Ranchi.

In Vedic literature the word *kha* often denoted the hole in the nave of a chariot-wheel. Recent studies indicate that Panini (about 500 BC) had some concept of zero for which a symbol was definitely used by Pingala (cf. of his rule *Rupe sunyam*) who is generally placed at about 200 BC. These concepts and notations combined with the age-old decimal numeration and the evolving Brahmi numerals were fused to invent in India the famous decimal place-value system of numerals about 2000 years ago. The exact time, place and name of inventor is not yet known, but it is certain that the system was already in use in India during the first centuries of the present era. Chinese had a positional system but they named the ranks. Babylonians also used the principle of position in their sexagesimal system and put some zero-symbol to partial use (about third century BC). But it was in India that a perfect and abstract decimal place-value system was developed. Due to prejudice and other factors the adoption of the simple, elegant, and perfect Indian system was slow in many parts of the world. But the world-wide victory of Indian numerals was almost complete when finally in the seventeenth century they were adopted universally. In vain Charles XII, King of Sweden (1682-1718) tried to abolish the Indian decimal system in favour of some other system. Of course, due to use in computer designs, the binary system has gained new significance and importance in modern times.

In *Rgveda* (the oldest of the four Vedas) X, 90, 4 occurs the word *tripada* which is taken to be the earliest example of a composite fraction (here 3/4). In the *Maitrayani Saṁhita,* III, 7, 7 are mentioned the unit fractions 1/4, 1/8, 1/12, 1/16, (= *kala*). The Babylonians (2nd millennium BC) used sexagesimal fractions, e. g. they expressed

$$\sqrt{2} = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}.$$

The sexagesimal system of fractions survives these days in the division of angular degree and hour into minutes and seconds. The Indian *Sulba sutra* value was expressed in unit fractions as

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}.$$

Unit fractions are often called “Egyptian fractions’ because of their extensive use in ancient Egypt. For example, the Rhind Mathematical Papyrus (about 1600 BC) contains a table of unit fractions for $2/n$ for each odd integer n from 3 to 101. A sample is

$$\frac{2}{43} = \frac{1}{42} + \frac{1}{86} + \frac{1}{129} + \frac{1}{301}.$$

In metrology the use of fractional quantities was avoided by dividing various units into sub-units of different orders. The Buddhistic work *Lalitavistara* divides an *angula* breadth (about 3/4 inches) into 7 *yavas*, each *yava* into 7 *sarsapas* and so on, getting finally one *angula* equal to 7^{10} *paramanu-rajas* (atomic particles). The Jainas divided an *angula* into 8^{12} *avasannasanna skandhas* (taking 8 subunits each time). Mahavira (850 AD) gave an extensive treatment of fractions including a rule for expressing any given unit fraction as sum of r fractions with given numerators a_1 to a_r. Fibonacci (thirteenth century) in his *Liber Abaci* described a technique for converting any given fraction into a sum of unit fractions.

Mathematically, any number of the form a/b, where a and b are integers $(b \neq 0)$ is called a rational number;otherwise it is called irrational. The ancient Greeks treated linear quantities and numbers as segments (of straight lines). They called two segments ‘commensurable’ if there was a segment which measured each of the given segments, that is contained exactly a whole number of times in each of the given segments. If two segments represented rational numbers, say m/n and p/q, they will be always, 'commensurable' (as each is a multiple of the segment represented by $1/nq$). Greeks discovered (about 500 BC) that the diagonal and side of any square were incommensurable with each other, and proved it (by showing equivalently that $\sqrt{2} \neq a/b$). The irrationality of the number π was first proved by J. H. Lambert (1728-1777):

The successful introduction of decimal fractions is usually assigned to Simon Stevin who set forth the system in his *La Disme* (1585), but their use was made earlier by the Arabs, e. g. al-kashi (about 1425). Stevin also advocated the use of decimal system of weights and measures but the idea did not become a reality until the French Revolution of 1789 after which the metric system has been slowly gaining ground. The fractional part of any number can be expressed in decimal fractions as

$$\frac{b_1}{10}+\frac{b_2}{10^2}+\frac{b_3}{10^3}+\ \ldots\ldots$$

which is written with a decimal point as $b_1b_2b_3...$(b's being 0, 1, 2, etc. up to 9). For rational numbers this decimal fractional part will either terminate or recur. For irrational numbers (e. g. $\sqrt{2}$, π) it will be unending (and *vice versa*). The totality of rationals (including 0 and integers both positive and negative) and irrrationals form the complete set of real numbers (or whole real axis).

Recognition and acceptance of negative numbers was slow especially among the Greeks. For example, Diophantus (third century AD) regarded the equation

$$4x + 20 = 4$$

to be absurd (since for him, negative numbers did not exist).

The first degree equation

$$px + q = 0, (p \neq 0)$$

where p and q are any integers gives the rational root $-q/p$ which represents the unique solution in all other cases also whether the root is rational or irrational, positive or negative or zero However, the equation

$$ax^2 + c = 0$$

presented great difficulties when a and c were of the like sign, since the solution demanded something new. It amounted to finding square-root of a negative number. The credit goes to the Indian mathematician Mahavira (850 AD) for stating in his *Ganitasara-sangraha* (I, 52) that

> ऋणं स्वरूप तोडवर्गो यतस्तस्मान्न तत्पदम्।
> (A negative number is nonsquare by its own form, therefore there is no (real) square-root from it.)

Although Cardan (1545) considered the problem of solving

$$x + y = 10 \qquad xy = 40$$

to be 'manifestly impossible', he nevertheless obtained the solution

$$x = (5 + \sqrt{-15}), \quad y = (5 - \sqrt{-15}).$$

But he discarded these numbers as 'useless'. Albert Girard (about 1625) accepted imaginary numbers and said that this will enable us to get as many roots (real or complex) as the degree of a given polynomial equation. Euler (1777) successfully used imaginary numbers and introduced the symbol i for $\sqrt{-1}$. W. R. Hamilton (about 1830) regarded the complex number $(a + ib)$ as an ordered pair of real numbers (a, b) thus giving a purely arithmetical definition and avoiding use of the so called 'imaginary' numbers (which are a very much reality in mathematics). Complex numbers were also interpreted as vectors (for addition and subtraction), as operators performing magnifications and rotations (by multiplication and division), and as a matix The relation

$$e^{i\pi} + 1 = 0 \tag{1}$$

is one of the most beautiful formula of elementary mathematics as it involves five epoch-making numbers (0, 1, i, e and π) in a simple way. It helped in finding the nature of π.

By considering a general polynomial equation

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$$

where the coefficients a_0, a_1, a_2, etc. are integers, a new classification of numbers was made. Any number, whether real or complex, which satisfied the above equation was called an 'algebraic' number, and a number which is not algebraic was called "transcendental". It is obvious that algebraic numbers include all rational numbers and many irrational and complex (or imaginary) numbers. The existence of transcendental numbers was demonstrated by Joseph Lioville in 1844. In general a number of the form

$$\sum_{n=1}^{\infty} \frac{c_n}{10^{n!}}$$

where c_1, c_2, c_3, etc. are arbitrary integers from 1 to 9, are found to be non-algebraic. These form of numbers are called 'Liouville's Numbers'. Investigations about transcendental numbers (defined as limits of certain continued fractions) were published by him in *Journal de mathematiques,* Vol. 16 (1851), 133-142.

In 1873 Charles Hermite proved, by working along the lines of Lioville, that the number Q (the base of natural logarithms) was transcendental (see *Comptes Rendus,* Vol. 77). By modifying the proof of Hermite, the German mathematician Carl Louis Ferdinand Lindemann (1852-1939) was able to prove the transcendence of the famous number π. The proof was based on the theorem that if A_r and c_r are algebraic numbers then (leaving aside the trivial cases) the expression cannot be

$$\sum_{r=1}^{n} A_r e$$

zero. The proof then follows due to (1). In this way π cannot be the root of an algebraic equation nor it is expressible in terms of radicals. Thus the quadrature (and rectification) of a circle by Euclidean tools (ruler and compass) is impossible. This settled (of course negatively) the 5000 year old problem of squaring the circle.

The existence of much more general transcendental numbers was proved by Georg Cantor (1845-1918) by using his theory of transfinite cardinal numbers in a paper published in the *Crelle's Journal*, Vol. 77 (1874). In fact this interesting proof demonstrated that there are 'more' transcendental numbers than algebraic (the cardinality of the set of algebraic numbers is just the same as that of positive integers).

But the problems in the theory of algebraic and transcendental numbers have been and are still a challenge to mathematicians. One such, proposed by David Hilbert in 1900, was to find whether $2^{\sqrt{2}}$ was algebraic or not. The question was settled when A. Gelfond (1934) and T. Schneider (1935) showed that a^b is transcendental where a is algebraic ($\neq 0$ or 1) and b is algebraic and irrational.

In our discussions we had taken the natural numbers for granted. Gottloob Frege in his *Foundation of Arithmetic* (1884) had said that the set of all sets which can be put into one-to-one correspondence with each other defines a cardinal (or natural) number Five years later G. Peano gave an axiomatic approach to define natural numbers assuming only the unity (i. e. number one) and a succession law. Kronecker has said that "God created integers, all the rest is the work of man" Pythagoreans had said that number one is God (and so it cannot be created by human beings). But given one, Peano could define other integers and the construction of the rest followed.

ALGEBRA

> If one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes, then the learned scholars of Hindustan are the real inventors of algebra.
>
> — H. HANKEL

The word 'algebra' is from the Arabic root 'al-jabr' which means adding equal terms on both sides (the Sanskrit name for algebra was *Bijaganita*).

The early (elementary) phase of algebra covers the period from about 2000 BC to 1700 AD. The development of algebraic notation programmed through the following three stages:

(i) Rhetoric (or verbal, with no symbology)
(ii) Syncopated (in which abbreviations of words were used, e.g. यावत्तावत्-वर्ग was shortened to 'याव' which denoted x^2 of modern time)*
(iii) Symbolic (during which full symbology was developed and used).

uadratic Equations

Problems involving quadratic equations are found profusely in ancient Babylonian mathematics. The terminology used was geometric but the methods of olutions were arithmetical, e.g. one problem reads:

"Length and width added are 14, and 48 is the surface." which is equivalent to olving of

$$x + y = a; xy = b$$

ith a and b equal to 14 and 48 here.

Babylonians found only the positive root in all such problems (by taking only ositive value of square-root involved)

---

vattavat is the ancient Indian term for the unknown. Thus 5 x^3 was written as याघ ५

Due to mathematical rigour, the Greeks faced conceptual difficulties with irrational numbers, and represented quantities by line segments and areas. For, example, the equation $x^2 + ax = b^2$ was solved geometrically.

The strict geometric formulations and methods were unwieldly; could not handle negative quantities, and hampered progress in the field of algebra. However, Diophantus (third century AD) who introduced syncopation in Greek algebra, followed Babylonian procedures.

Problems leading to quadratic equations are found in *Sulba Sutras* (c. 500 BC). The Chinese work *Chiu Chang Suan Shu* (first century AD) has a problem which was solved by finding the positive root of an equation of the type $x^2 + ax = b^2$

Aryabhata I (born 476 AD) in his *Aryabhatiya* (II, 20) has given a rule for finding the number of terms, n, of an arithmetical progression when its first term, common difference, and sum are known. This amounts to solving the equation

$$dn^2 + (2a - d)\, n = 2s$$

Brahmagupta (628 AD) has given a general rule for finding a root of the equation

$$ax^2 + bx = c$$

However, it was Sridhara (about 750 AD) who became popular for his rule:

चतुराहतवर्गसमैः रूपैः पक्षद्वयं गुणयेत्।
अव्यक्तवर्ग रूपैयुक्तौ पक्षौ ततो मूलम्।।

which states that the equation ($ax^2 + bx = c$) should be multiplied by $4a$, and b^2 should be added to both sides; and then square-roots should be extracted. That is, we get

$$4a^2x^2 + 4abx = 4ac$$

then $$(2ax + b)^2 = 4ac + b^2$$

and, finally, the solution will follow after extracting of square-roots, etc. To give a single rule for all quadratic equations is a significant contribution of India. The famous Arab algebraist al-Khwarizmi (ninth century) dealt separately with

$ax^2 + bx = c,$

$ax^2 + c = bx$

and $x^2 = bx + c$

taking a, b, c as positive.

Vieta (1540-1603) gave a new method for solving

$x^2 + ax + b = 0$

He put x equal to $(u + z)$ and got

$u^2 + (2z + a)\, u + (z^2 + az + b) = 0$

Choosing $(2z + a)$ to be zero, he reduced the above to

$u^2 = (a^2 - 4b)/4$

thereby getting u, and hence x (both roots). Harriot (1631) seems to be the first who solved the equation

$$x^2 - bx + cx = bc$$

y factorising as $(x - b)(x + c) = 0$ but still neglected the factor $(x + c)$.

Relations between coefficients and roots were given by A. Girard (1629). Glimpse of omplex roots ($5 \pm \sqrt{-15}$) were found by Cardan (1545) for the equations $x + y =$, $xy = b$ with $a = 10$ and $b = 40$, although the mysterious roots caused him mental orture.

It is interesting to note that for solving a general quadratic equation

$ax^2 + bx + c = 0$

Swami Bharati Krishnatirthaji (about 950) gave the *sutra*

चलितकलित वर्गो विवेचक:

r "(Differential coefficient)^2 = Discriminant" which will immediately yield

$(2ax + b)^2 = b^2 - 4ac$

olution then follows by taking square-roots similar to what was prescribed by ridhara.

ermutations and Combinations

Relevant material on basic combinatorial lore in Classical Greek has not come to ght. Chrysippus (third century BC), a Stoic philosopher, simply stated that the umber of compound propositions which can be made from ten simple propositions xceeds a million (he could not give the exact number which is 2^{10}).

In India, the Jainas took special interest in the subject by including it among athematical topics. In their canonical literature (about 300 BC), permutations and mbinations were called *Vikalpa-ganita* and *Bhanga* respectively. The *Bhagavati utra* discusses different philosophical categories which can arise out of the combinan of n fundamental categories taken one at a time *(eka-samyoga)*, two at a time *vikasamyoga)*, etc. The author could also find the correct values of $^nC_1$, $^nC_2$, etc. d of $^nP_1$, $^nP_2$, etc.

The theory of permutations and combinations is extensively dealt in works on trical science of which Pingala's *Chandah-sutra* (about 200 BC) is a basic and ndard ancient work. In the *varna-vrttas* (meters based on letters) of Sanskrit osody, the number of letters remain constant. The *laghu* letter has one mora atra or syllable instant) and the *guru* letter has two morae. They are denoted by symbols I, and S respectively. The exapansions giving all the possible variations various *varna-vrttas* can be tabulated as follows:

Table of varna-vrttas

1 letter	*2 letters*	*3 letters*	
(1) S	(1) S S	(1) S S S	(5) S S I
(2) I	(2) I S	(2) I S S	(6) I S I
	(3) S I	(3) S I S	(7) S I I
	(4) I I	(4) I I S	(8) I I I

The total number of meters of this category up to n letters will be

$$2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots + 2^n$$

Varahamihira(sixth century AD) in his *Brhotsamhita* (76, 17) has given the total number of perfumes which can be prepared from 16 ingradients when mixed in different proportion from one to four (the number mentioned is 174720). The *Ganitasara-sangraha* of Mahavira gives the general formula for calculating ^nC_r. Silanka (about 862) in his commentary on Sutrakrtanga gives $^nP_n = 1.2.3.......n$. Values of ^nC_r for various values of n and r $(= 0,1,2, ...,n)$ are found in the Meruprastara mentioned by Halayudha (tenth century) and also in the Suciprastara. Bhaskara II (1150 AD) gave the name *ankapasa* to the subject of permutations and combinations. He has dealt with general formulas for ^nC_r, ^nP_r and for arrangements of objects which includes similar ones also. Narayana Pandita (1356) in his *Ganitakaumudi* has given advanced treatment of the various type of ankapasas ('Net of Numbers'), e.g. rule for finding the number of variations in which a digit d occurs t times when all possible arrangements of numbers, having a total of p digits each, are written with q digits $(< p)$ with repetitions.

Tartaglia (1623 AD) seems to be the first to have applied the theory of permutations and combinations to the throwing of dice. In Buteo's *Logistica* (1560) is found a discussion of the problem of a combination lock with several movable cylinders (it will open only when all the cylinders have definite positions known to the owner).

Formulas for permutations and combinations in full generality and employing modern notation occur in Herigone's *Cursus Mathematicus* (1634).

A problem which is related to arrangement of objects in a circle is the famous Josephus Problem which is found in one form or the other in western as well as oriental literature, e.g. in even the late Japanese work called *Shojutsu Sangaku Zuye* (1795). A similar problem is found in Buteo's *Logistica*. It required to arrange 15 Turks and 15 Christians in a circle so that, in counting round, every 15th one should be a Turk.

Jacques *Bernoulli's Ars Conjectandi* (posthumously published in 1713) is an extensive work on the subject as taught at elementary level these days. In it appears the word 'permutation' with present meaning. Earlier, Leibniz used the word 'variationes', and Wallis had adopted 'alternationes' which is literal translation of the Sanskrit word *vikalpa* used 2000 years earlier in India for the same purpose! The new symbol n ! for factorial n was introduced by C. Kramp in 1808 to facilitate printing (and typing), and has now almost completely replaced the older awkward form n.

Sequences and Series

Simple sequences such as 1, 3, 5, ... 33 and 4, 8, 12, ... 48, are found in the *Yajurved* (second millennium BC). The *Pancavimsa Brahmana* (XVIII, 3) contains the sequence 12, 24, 48, ... 196608, 393216, and the Buddhist work *Digha*

Nikaya uses the sequence 10, 20, 40,............................,80000 in calculation of increment in human life. The Sanskrit work *Brhaddevata* (about 400 BC) gives the result

$$2 + 3 + 4 + \cdots\cdots + 1000 = 500\,499$$

and the Jaina work *Kalpa Sutra* of Bhadrabahu (about 300 B C) has

$$1 + 2 + 4 + \cdots\cdots + 8192 = 16383.$$

Sequences of numbers forming geometrical progressions with common ratio 2 are also met with in the description of lands, mountains, and seas in and around Jambudvipa of Jaina cosmography. Of course, the numbers 2^n also occur in connection with the total number of arrangements of n places with elements of a binary set. It is unfortunate that a Jaina work, called *Brhaddhara Parikarma,* on series is lost.

Greeks were familiar with arithmetical progressions but they were more interested in the figurate numbers, namely, triangular numbers: 1, 3, 6, 10, ...; square numbers: 1, 4, 9, ...; pentagonal numbers: 1, 5, 12, 22, 35, ..., etc. They are also called polygonal numbers as they can be set in geometrical patterns forming polygons; Nicomachus of Gersa (about 100 AD) extensively dealt with these numbers in his *Introductio Arithmetica.*

Correct formulas related to the sum of any A. P. are given by Aryabhata I along with the following results (Aryabhatiya, II 19-22):

$1 + 3 + 6 + \ldots\ldots$ upto n terms $= n(n + 1)(n + 2)/6$

$1^2 + 2^2 + 3^2 + \cdots\cdots + n^2 = n(n + 1)(2n + 1)/6$

$1^3 + 2^3 + 3^3 + \cdots\cdots + n^3. = (1 + 2 + \cdots\cdots + n)^2$

Sridhara (ca. 750 AD) has given rules for sums of series of terms which are squares or cubes of an A.P. By following the method of *varna-samvargita*, Virasena (816 AD) got the following sequence from 2:

$2^2\ (= 4), (4^4)\ (= 256), 256^{256}$, etc.

The usual rule for the sum of a G. P. is found in the Tiloyapannatti (III, 80). A geometrical demonstration of the rule occurs in the *Kriyakramakari* commentary (sixteenth century) on bilavati.

The name of Fibonacci (1170-1250) is attached with the sequence 1, 1, 2, 3, 5, 8, 13,............ where $u_{n+2} = u_{n+1} + u_n$. But the sequence was well known in India much before Fibonacci's time, in connection with metrical science *(Chandah-sastra or prosody)*. Sanskrit and Prakrit meters of the type called *matra-vrttas* are those in which the number of morae remain constant. Usually a *laghu* letter has one mora (matra), and a *guru* letter has two morae (they are denoted by I and S respectively). The table of expansions giving variations in formation of *matravrttas* will be as follows.

Table of matravrttas

1 morae	2 morae	3 morae	4 morae	5 morae	
I	S	I S	(1) S S	(1) I S S	(5) I I S I
	I I	S I	(2) I I S	(2) S I S	(6) I S I I
		I I I	(3) I S I	(3) I I I S	(7) S I I I
			(4) S I I	(4) S S I	(8) I I I I I
			(5) I I I I		

Thus we see that the numbers of variations of meters having 1, 2, 3, ... mora are the so-called Fibonacci numbers. A knowledge of these numbers is indicated in Pingala's work (ca. 200 BC). Rules for forming Fibonacci sequence are explicitly given by Virahanka (between 600 and 800 AD), Gopal (before 1135), and Hemacandra (ca. 1150).

Nemicandra (10th century) has mentioned 14 types of sequences some of which are unusal e. g. the dvirupa-ghana sequence is $2^{3.2^0}$, $2^{3.2^{\prime}}$, $2^{3.2^2}$, $2^{3.2^3}$, .. etc.

It is said that his *Trilokasara* is the only work which gives information about the "dyadic type of sequences which pass through various types of finite and transfinite cardinals of sets". Narayana (1356) has given rules for the sum of series whose terms are formed successively by taking the partial sums of other series.

Archimedes (ca. 225 BC) seems to be the first to sum an infinite G. P. In India Virasena (816 AD) used an infinite process in connection with computation of volume of a frustum of a cone. Stifel (1544) considered the "astronomical progression".

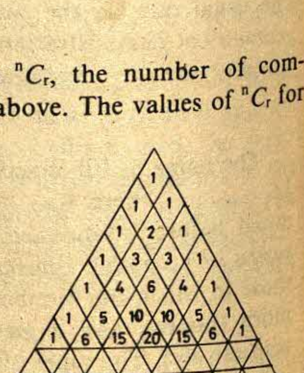
$$1 + 1/60 + 1/60^2 + 1/60^3 + \ldots\ldots$$

The general formula for the sum of an infinite G. P. was given by Vieta in 1590. It may be mentioned that the Greek name for a series as used by the Pythagoreans was *ekthesis* (literally a selling out). Boethius (sixth century AD), like other Latin writers used the word 'progressio'. It was only much later on that the name 'series' was used, e. g. by James Gregory in 1671.

Binomial Theorem

A historical discussion about the rules for finding ${}^nC_r$, the number of combinations of n things taken r at a time, is already given above. The values of ${}^nC_r$ for different n are as follows:

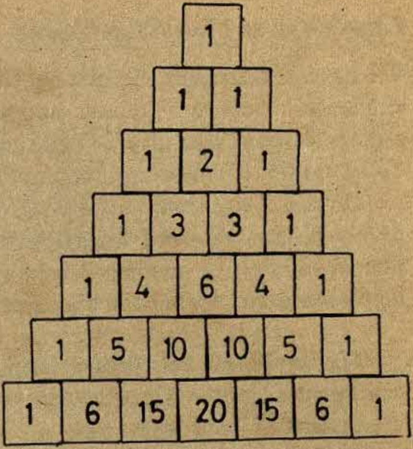
For $n = 1$, ${}^1C_0 = 1$, ${}^1C_1 = 1$;
for $n = 2$, ${}^2C_0 = 1$, ${}^2C_1 = 2$, ${}^2C_2 = 1$;
for $n = 3$, ${}^3C_0 = 1$, ${}^3C_1 = {}^3C_2 = 3$ ${}^3C_3 = 1$;
etc. If these are arranged symmetrically in rows one below the other we shall get a triangular pattern as shown here (for completeness, we have put an extra 1 at the top, and used squarically in ancient Indian style.)



This arrangement is commonly known as Pascal Triangle after Blaise Pascal who called it Arithmetical Triangle and gave it in a work which was published posthumously in 1665.

In India the triangular arrangement has been vividly described by *Halayndha* (tenth century) in his Sanskrit commentary on a *sutra* (VIII, 34) of Pingala's work on prosody.. It is called *Meruprastara*. The numbers in the $(n+1)$th row of the array gives the values of the combination numbers

${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots\ldots\ldots, {}^nC_r, \ldots, {}^nC_n$ (A)



The Ratna-manjusa (about 900 AD), a work on sanskrit prosody, contains another method of getting these combination numbers (A). The method is called *Suci-prastara*, and there is mentioned still another process called *khandameru* (ascribed to Punnagacandra) for the same purpose.

The most significant and useful thing about the combination numbers (A) is that they occur also as coefficients of various terms in the expansion of $(a+b)^n$ by binomial theorem. That is

$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}\, b^2 + \ldots\ldots + {}^nC_n\, b^n$ ..... (B)

Expansion of $(a+b)^n$ for smaller positive integral values of n were already known earlier (as they can be found by multipication), but the recognition and identification of the combination numbers (A) as binomial coefficients in the *general case* (B) will definitely imply a knowledge of the binomial theorem for positive integral index. It is difficult to name the identifier.

Among the Arabs, Umar Khayyam (about 1130) claimed to have discovered the law of expansion of $(a+b)^n$, but **al-Karaji** may have known the formula a century earlier. Al-Zanjani (d. 1262) used it for $n = 7$. Although binomial theorem for positive integral power was known in the East long before its appearance in the west, it was Pascal's work (1665) which made Europe familiar with it.

The generalisation for negative and fractional index is due to Isac Newton who stated it (without proof) in two letters which he wrote in 1676 to Henry Oldenberg. The first proof (but not rigorous) for arbitrary positive integral power seems to be by Jakobe (or Jacques) Bernoulli whose work was published posthumously in 1713. Maclaurin (1742) and Euler (1774) proved it for rational or fractional index. In 1826 N. H. Abel, then only 24 year old, published the first general proof for any complex exponent.

Exponential and Logarithmic Series

We have already pointed out that the numbers 2^n and the G. P. formed by them are extensively needed in Jaina cosmography. To handle these numbers the Jaina mathematicians had invented and used logarithms to base 2 quite early. Virasena in his *Dhavala* commentary (about 816 AD) has given rules equivalent to modern formulas for log (mn), log (m/n), log m^n, etc. Logarithm to base two was called *ardhaccheda*. The word 'logarithm' is formed from two Greek words *logos* (= ratio) and *arithmos* (= number), the corresponding phonetically similar Indian word *laghuriktha* for it was coined by Sudhakara Dvivedi (about 1900 AD).

The invention (or rather reinvention) of logarithms in Europe is generally attributed to John Napier (1550-1617). However, when Napier first described his logarithms in 1614, they were not dependent upon the idea of a base. But in an appendix to the English translation (1618) of Napier's original Latin work, there is a table of natural logarithms (i. e. the base is the number e).

Leonhard Euler's *Introductio in Analysis Infinitorum* (1748) deals extensively with the number e (In fact this symbol is also due to him). He stated that the number e is given by the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \qquad \ldots\ldots \text{(1)}$$

whose sum can be shown to be an irrational number lying between 2 and 3. By considering the binomial expansion

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \left(1 - \frac{1}{n}\right)\cdot\frac{1}{2!} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdot\frac{1}{3!} + \ldots\ldots \text{(2)}$$

and making n limitlessly large, the right hand side of (2) will be same as series (1) Similar limiting process yields the series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

from the binomial expansion of $\left(1 \mp \frac{1}{n}\right)^{nx}$ Combining all these resulted in getting the exponential series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

Using the relation $e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$

Euler calculated e to 23 decimal places (2.71828...). He also gave two continued fraction expansions for it. The transcendence of e was proved by Charles Hermite in 1873. In 1926, D. H. Lehmer computed the value of e to 709 decimal places by using a continued-fraction expansion.

The idea of expressing a logarithm by means of a series seems to have originated with James Gregory. Newton (born 1642) also saw that it was possible to express log $(1 + x)$ as an infinite series by expanding $1/(1+ x)$ as

$$1 - x + x^2 - x^3 + \ldots\ldots$$

either by long division or by binomial expansion, and then applying rule of quadrature (or integration) to each term. However, it was Nicolaus Mercator who first published in 1668 the series

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}\text{.........}$$

The expansion of $(1/2)\log[(1+x)/(1-x)]$ was found by John Wallis in 1695. R. Cotes stated in his *Harmonia Mensurarum* (published posthumously in 1722) that

$$\log(\cos\theta+i\sin\theta)=i\theta$$

which is equivalent to

$$e^{i\theta}=\cos\theta+i\sin\theta$$

a result which bears Euler's name but which was already known to De Moivre (perhaps as early as 1707).

Mathematical Induction

Like many of the concepts and methods of mathematics, proof by mathematical induction is not the invention of a particular individual at a fixed moment. Rather, the method is implicit, in some form or the other in the works of several early mathematicians; and, after gradual evolution, it slowly emerges until at last a satisfactory, fully explicit formulation is given at a later date.

Processes akin to mathematical induction, some of which would yield the modern mathematical induction by introducing some slight change in the mode of presentation or in the point of view, were given long ago. One such process was the recurrent mode of inference by regular progression from a particular value of n to next. In essence the principle of mathematical induction was known to the Pythagoreans (sixth century BC). It is implicit in Euclid's proof (*Elements*, IX, 20) that the number of primes is infinite, in Proclus' process of finding numbers representing the sides and diagonals of squares (fifth century AD), in the Indian cyclid method (*cakravala*) of solving indeterminate equations (about 800 AD) in al-Karaji's (ca. 1030) demonstration of $(a+b)^n$ for $n=3, 4$, and in Campanus' proof (thirteenth century) of the irrationality of the golden ratio, $(\sqrt{5}-1)/2$.

The discovery of the inference by mathematical induction in the modern sense is ascribed to Francesco Maurolico who was the first to make a fairly explicit use of the method in a work published in 1575. However, it was not until the seventeenth century that satisfactory formulations of the method are found in the work of Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662).

Fermat used his version of mathematical induction or reasoning by recurrence (also called Fermatian Induction) to prove that there are no positive integers x, y, z such that $x^3+y^3=z^3$. Going further, he stated the general proposition that for n an integer greater than 2, there are no positive integral values x, y, z, such that $x^n+y^n=z^n$ (This is the famous Fermat's Last Theorem). Pascal had used the method of mathematical induction very explicitly and repeatedly in his work on the Arithmetical

Triangle e.g. in proving the relation

$$\frac{{}^{n}C_{r}}{{}^{n}C_{r+1}} = \frac{r+1}{n-r}$$

The name 'mathematical induction' seems to have originated much later in Augustus De Morgan's article on 'Induction (Mathematics)' in *Penny Cyclopaedia* of 1838. The name and method were adopted and popularised by I. Todhunter (died 1884) in his *Algebra.* It should be noted that in natural science also there is 'induction' (called Baconian Induction). Today, the Baconian induction is called 'incomplete induction' and the mathematical induction is called 'Complete induction'. In fact, the modern mathematical induction is a rigorous method of deduction.

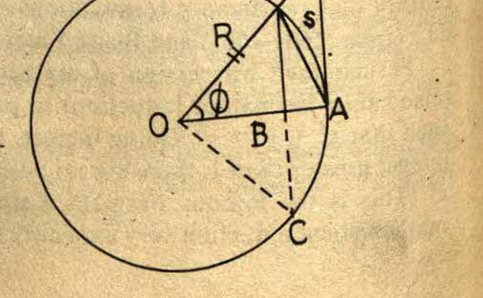
TRIGONOMETRY

> Thus was born, apparently in India, the predecessor of the modern trigonometric function known as the sine of an angle; and the introduction of the sine function represents the chief contribution of the *siddhantas* (Sanskrit astronomical works) to the history of mathematics.
>
> CARL B. BOYER (historian of mathematics)

The science of trigonometry was cultivated, in the beginning, primarily for the purpose of perfecting astronimical calculations, by assuming the round figure of 12 months, of 30 days each, in a year, the ecliptic (apparent circular path of the sun) was divided into 360 parts. The idea of dividing a circle (*cakra*) into 12 bigger parts and 360 smaller parts is mentioned in *Rgveda* (I, 164, 48). The full system of dividing the circumference of a circle (or the central angle) into angular units, as developed later on, was as follows:

1 circle (*cakra*) = 12 signs (*rasis*)
= 360 degrees (*amsa*) =21600 minutes (*katas*)
= 360×60^2 seconds (*vikalas*) = 360×60^3 thirds (*tatparas*).

Let R be the radius of the circle of reference. In ancient times the argument (independent variable) corresponding to any point P on the circle was the arc PA $(=s)$ measured from a point A. The arc was generally measured in the same units as the angle ϕ subtended by it at the centre. In Greek trigonometry the funcion considered was the chord PA of the arc s.



In contrast to Greek trigonometry of chord-function, the Indian trigonometry was based on the sine function. The Indian Sine (written with a capital S) of the arc $PA(=s)$ is difined by the length PB which is the perpendicular from P on the radius OP through the initial or reference point A. Since the length is also half of the chord PC, the Indian Sine was called *ardhjya* or *ardhajiva* originally. Often and later on the perfix *ardha* was dropped. The modern sine of the angle ϕ (written as $\sin \phi$) is defined either by the ratio PB/OP in the right triangle OPB, or by the length PB when R is taken unity. The relationship between the Greek chord function and the Indian Sine function can be easily seen to be

$$\text{Sin } S = (\text{crd } 2S)/2 = R \sin \phi$$

The other three ancient Indian trigonometric functions were the Cosine (defined by the length OB), the Versed Sine (defined by the length BA), and its complement. In modern times we define six trigonometric functions namely, sine, cosine, tangent (which is defined either by $\sin \phi/\cos \phi$, or by PB/OB, or by the length of the tangent AT when $R = 1$) and their reciprocals.

These functions (the most basic of which is the Indian Sine) drove away the Greek chord-function for ever just as the Indian numerals triumphed over other systems. In fact, modern trigonometry based on sine and its associates is another significant gift of India to world mathematics. In fact the English mathematical world 'Sine' itself was generated from The Sanskrit word Jiva (जीवा) via The Arab jyle (   ) and the Latin *Sinus* in an interesting manner.

During the medieval period (fifth to seventeenth century) when trigonometry based on Indian Sine, Versed Sine (*utkramajya*), etc. was being used all over the scientific world, various values for the parameter R (called *sinus totus*) were chosen in India and abroad such as 3438 (Aryabhata I, Bhaskara I and II, and many others), 150 (Brahmagupta, al-Fazari, etc.), 12375859 (Govindasvamin and Udayadivakara), 10^7 (Regiomontanus and Rheticus), 10^{10} (Otho and Romanus), etc. Aryabhata's value was derived from the formula

$$\begin{aligned} R = C/2\pi &= 360 \times 60/6.2832 \\ &= 57^\circ \text{ or } 3438' \text{ nearly.} \end{aligned}$$

This choice brought the Indians very near to the definition of the radian measure of angle.

The Arab Abu'l Wafa (about 980 AD) is said to be the first to choose $R = 1$, and Rheticus (1550) was the first European to discard the arc and use trigonometric functions as ratios of sides in right triangle, thus making them pure numbers (free from units). However, sines and cosines continued to be taken as lines up to quite later times, e.g. by Lacroix (about 1800). The Indian word *koti–jya* ('Sine of complementary arc') is shortened to *kojya* and resembles the formation of the term 'cosine' which emerged from a suggestion by Edmund Counter (1620) to write sine of the complement angle as Co. sinus.

For finding the sine of an arc or angle greater than 90°, the method used in

India (from fifth century onwards) was to apply the following formulas ($\phi < 90°$)

$$R \sin (90+\phi) = R - R \text{ vers } \phi = R \sin (90 - \phi)$$
$$R \sin (180+\phi) = - R \sin \phi$$
$$R \sin (270+\phi) = R + R \text{ vers } \phi = - R \sin (90 - \phi)$$

These formulas have been given by Bhaskara I (about 600 AD). In these the values of R sin ϕ and R vers ϕ for $\phi < 90°$, were obtained with the help of tables when necessary. Varahamihira (about 550) gives the equivalent of the identities

$$(R \sin \phi)^2 = (R/2)\,[R - R \sin (90° - 2\phi)]$$

$$= \left[\frac{R \sin 2\phi}{2}\right]^2 + \left[\frac{R - R\sin (90° - 2\phi)}{2}\right]^2$$

Using these, sines of $A/2^n$ were found from sin A, and this 'bisection method' was employed in preparing sine tables. Formulas equivalent to

$$\sin (A + B) = \sin A . \cos B + \cos A . \sin B$$

were obtained by Abu'l Wafa. Bhaskara II (1150) called them Samasa and Antara Bhavana. A sixteenth century Malayalam work Yuktibhasa contains a proof for the expansion of sin (A + B).

For remembering sines of simple angles, the 'telegraphic' rule of Brahmagupta (628 AD), with slight change, is

$$\sqrt{(0, 1, 2, 3, 4)/4} = \sin (0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2})$$

Exact expressions for sines or cosines of 18°, 36°, 54°, 72°, etc. are given by Bhaskara II. The value of sine of 24° was specially mentioned in Indian astonomical works because 24° was assumed to be the obliquity of the ecliptic. The following marvellous approximation formula is found in all important Indian works starting with the *Mahabhaskariya* of Bhaskara I

$$\sin \phi = 4\phi\,(180 - \phi) / [40500 - \phi\,(180 - \phi)]$$

where ϕ is in degrees.

The *Aryabhatiya* of Aryabhata I (b. 476 AD) is the earliest *pauruseya* work which gives table of Sines (R = 3438, and h = 225 minutes). Their 24 differences from *makhi* (= 225) to *cha* (= 7) are described just in one couplet. By 1613 Pitiscus had published table of sines and cosines correct to 15 decimal places. In his auto-commentary on *Jyotpatti*, Bhaskara II has outlined the method of finding sines of multiple angles, e.g., he says that we can get sin 5 ϕ by taking $A = 2\phi$ and $B = 3\varnothing$ in the sin $(A + B)$ formula. Vieta was able to express sin nx and cos nx in terms of powers of sin x and cos x. Due to his skill in this direction he could solve an equation of 45th degree which was set as an open challenge by Romanus. Rules for finding sines of submultiple angles (up to $A/5$) are found in the works of al-Kashi, Munisvara, and Kamalakara.

The rule (giving area of a triangle)

$$S = \sqrt{s(s-a)(s-b)(s-c)}$$

is called Heron's (or Hero's) formula. But al-Biruni states that it was already known to Archimedes (ca. 225 BC). However, the geometrical proof (i.e. without using any trigonometry) of the rule as given by Heron of Alexandria (ca. 75 AD) in his *Metrica* is the earliest one available to us. The more general formula

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

for the area of a cyclic quadrilateral was first given by Brahamagupta (628 AD). According to C.B. Boyer, the generalisation is perhaps "the most beautiful result" in Brahmagupta's work. When $d = 0$, it will reduce to Hero's rule. Brahmagupta also knew the rule (BSS, XII, 27)

$$\text{altitude } h = ab/2R$$

which immediately yields, for any triangle,

$$S = abc/4\,R$$

where R is circumradius. Regiomontanus gave

$$S = \frac{1}{2}\,ab\,\sin C$$

The symbols $\sin^{-1}x$, $\cos^{-1}x$, etc. for arc sin x, are cos x etc. were suggested by the astronomer Sir John F.W. Herschel (1813). The name of Thales (about 600 BC) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and of an auxiliary staff (or gnomon) of known height, and comparing the ratios

$$H/S = h/s \doteq \tan\text{ (sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works, e.g. *Aryabhatiya*, II, 16 is about finding height of an inaccessible lamp-post by measuring gnomon-shadows at two places. Recreational type of problems such as Hawk-and-rat problem (which was later on called Peacock-and-serpent problem), are found frequently. In medieval Europe, trigonometry was used in military problems, e.g. in finding range and correct elevation of artillery guns.

Without going into details, it may be pointed out that series for sin x, cos x and $\tan^{-1}x$ were known and proved in India two or three centuries before their appearance in the West. In fact, the subtle discussions show that seeds of modern analysis were sowed in India. The history of trigonometry exhibits within itself the embryonic growth of three classical divisions of mathematics: algebra, geometry, and analysis.

ANALYTIC GEOMETRY

Several stages involving various steps can be mentioned in the development of the analytic geometry such as (i) the correlation of number with geometrical magnitude; (ii) the invention of a system of coordinates; (iii) the recognition of a one-to-one correspondence between algebra and geometry; (iv) the graphical representation of the relation involving dependent and independent variables; and (v) the determination of the corresponding equation of the loci or curve. Some of the ideas involved in these steps belong to ancient, some to medieval and some to modern periods. According to S. Gunther, whenever we find an example of the stage (v) without others, especially (ii) and (iii), we may recognise a stroke of genius on the part of the scholar, but no analytical geometry.

All principal streets in Mohenjo-daro of the Indus Valley (about 2500 BC.) were well aligned and oriented towards the points of the compass. They intersected at right angles dividing the city into square or rectangular blocks. Perhaps they were laid out with the help of two rectangular axes. Primitive system of coordinates were used by the Egyptian Nilotic surveyors as early as 1400 BC.). In fact the districts (*hesp*) into which Egypt was divided were designated in hieroglyphics by a symbol which was derived from the pictograph of a grid (⊞).

Hipparchus (about 140 BC.) located points on the celestial and terrestrial globe by means of *mekos* (length or longitude or east-west distance) and *platos* (width or latitude). Romans laid out their towns with respect to two axes called *decimanus* (running east-west) and *cardo* (perpendicular to the above). To conceive position in space Vacaspati (ninth century) took three axes, one proceeding from the point of sunrise in the horizon to that of sunset; a second bisecting this line at right angles on the horizontal plane; and the third proceeding from the point of their section up to the meridian position of the sun on day. By this method (which anticipated coordinate geometry in India, according to B.N. Seal) Vacaspati indicated the position of one atom in space with reference to another.

S. Gunther would recognise three stages in the development of the idea of modern coordinate system: (a) Introduction of two axes: (b) Plotting of curves; and (c) Use of equations permitting one to go from abscissa to ordinate and *vice versa*. During the tenth and eleventh centuries, the paths of planets using 30 subdivisions in longitude and 12 in lattitude were plotted on a crude system of rectangular coordinates.

Graphical representation of function (as distinct from algebraic expressions) was first successfully done by Nicole Ovesme (about 1360). He used the terms 'latitudo' and 'longitudo' instead of ordinate and abscissa. Some historians have hailed him even as the creator of analytic geometry.

Modern analytic geometry is called 'Cartesian' after the name of Rene Descartes (1596-1650) whose relevant *La Geometrie* was published in 1637. But the fundamental principle and method of analytic geometry were already discovered by Pierre de Fermat (1601-1665) earlier (latest by 1636). Unfortunately, Fermat's treatise on the subject, entitled *Ad Locus Planos et Solidos Isagoge* ("Introduction to Plane and Solid Loci") was published only posthumously in 1679 (however the work was mentioned by Fermat in his correspondence). So Descartes came to be regarded as the unique inventor of the analytic geometry.

The *La Geometrie* of about a hundred pages was published as the third appendix to Descartes' *Dscours de la Methode* (1637). His procedure in the *Geometry* was to begin with a geometrical problem, to convert it into an algebraic equation, simplify it, and then solve the equation geometrically. Thus the work is devoted to geometry and vice versa.

Descartes took oblique ordinates for granted. Hence the usual formulas for distance, slope, angle between two lines, etc. are not to be found in his work. He uses letters x and y for abscissa and ordinate, although he did not use those words. He also did not formally introduce the y-axis. He emphasised more on derivation of equations of loci (or curves) while Fermat emphasised study of properties of curves given by equations.

The name 'Cartesian Geometry' to modern analytic geometry is an unfortunate title (because he was not the sole inventor,but it does justice to the fact that it was predominantly under the influence of Descartes that the new method took roots. Moreover, the date of his invention may be 1928 (or earlier) when he wrote a letter to Issac Beekmann. The new geometry did not bring about rapid changes or drastic transformation of mathematics quickly. But progress continued slowly. Explicit formulas were added and refinements were done.

Wallis (1965) considered conics 'absolutely'.(i.e. having nothing to do with cone) by giving purely analytic definitions and algebric treatment. Isaac Barrow disdained to use Cartesian method. Newton used method of unde termined coefficients to find equations of curves. He used several types of coordinates including polar and bipolar. Leibniz used the terms 'abscissa' 'ordinate' and 'coordinate'. L'Hospital (about 1700) wrote an important textbook an analytical geometry.

Clairut (1729) was the first to give the distance formula although in clumsy form. He also gave the intercept form of the linear-equation. Cramer (1750) made formal use of the two axes and gave the equation of a circle as

$$(y-a)^2+(b-x)^2=rr$$

He gave the best exposition of the analytic geometry of his time.

Monge (1781) gives the modern 'point-slope' form as

$$y-y'=a\,(x-x')$$

and the condition of perpendicularity of two lines as $aa' + 1 = 0$. S.F. Lacroix (1765-1843) was a prolific textbook writer, but his contributions to analytic geometry are found scattered. He gave the 'two-point' form as

$$y-\beta=\frac{\beta'-\beta}{\alpha'-\alpha}\;(x-\alpha)$$

and the perpendicular from (α, β) on $y = ax + b$ as
$(\beta - a\alpha - b) / \sqrt{1 + a^2}$

His formula for finding angle between two lines was $\tan \theta = (a' - a)/ (1 + aa')$
It is, of course, surprising that one has to wait for more than 150 years after the invention of analytic geometry before finding such essentially basic formulas. In 1818 C. Lame', a civil engineer, gives $mE + m'E' = 0$ as the curve passing through two loci $E, E' = 0$.

TABLES

TABLE I

Four-Place Values of Trigonometric Functions
Angle θ in Degrees and Radians

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
0° 00′	.0000	.0000	No value	.0000	No value	1.000	1.0000	1.5708	90° 00′
10	029	029	343.8	029	343.8	000	000	679	50
20	058	058	171.9	058	171.9	000	000	650	40
30	.0087	.0087	114.6	.0087	114.6	1.000	1.0000	1.5621	30
40	116	116	85.95	116	85.94	000	.9999	592	20
50	145	145	68.76	145	68.75	000	999	563	10
1° 00′	.0175	.0175	57.30	.0175	57.29	1.000	.9998	1.5533	89° 00′
10	204	204	49.11	204	49.10	000	998	504	50
20	233	233	42.98	233	42.96	000	997	475	40
30	.0262	.0262	38.20	.0262	38.19	1.000	.9997	1.5446	30
40	291	291	34.38	291	34.37	000	996	417	20
50	320	320	31.26	320	31.24	001	995	388	10
2° 00′	.0349	.0349	28.65	.0349	28.64	1.001	.9994	1.5359	88° 00′
10	378	378	26.45	378	26.43	001	993	330	50
20	407	407	24.56	407	24.54	001	992	301	40
30	.0436	.0436	22.93	.0437	22.90	1.001	.9990	1.5272	30
40	465	465	21.49	466	21.47	001	989	243	20
50	495	494	20.23	495	20.21	001	988	213	10
3° 00′	.0524	.0523	19.11	.0524	19.08	1.001	.9986	1.5184	87° 00′
10	553	552	18.10	553	18.07	002	985	155	50
20	582	581	17.20	582	17.17	002	983	126	40
30	.0611	.0610	16.38	.0612	16.35	1.002	.9981	1.5097	30
40	640	640	15.64	641	15.60	002	980	068	20
50	669	669	14.96	670	14.92	002	978	039	10
4° 00′	.0698	.0698	14.34	.0699	14.30	1.002	.9976	1.5010	86° 00′
10	727	727	13.76	729	13.73	003	974	981	50
20	756	756	13.23	758	13.20	003	971	952	40
30	.0785	.0785	12.75	.0787	12.71	1.003	.9969	1.4923	30
40	814	814	12.29	816	12.25	003	967	893	20
50	844	843	11.87	846	11.83	004	964	864	10
5° 00′	.0873	.0872	11.47	.0875	11.43	1.004	.9962	1.4835	85° 00′
10	902	901	11.10	904	11.06	004	959	806	50
20	931	929	10.76	934	10.71	004	957	777	40
30	.0960	.0958	10.43	.0963	10.39	1.005	.9954	1.4748	30
40	989	987	10.13	992	10.08	005	951	719	20
50	.1018	.1016	9.839	.1022	9.788	005	948	690	10
6° 00′	.1047	.1045	9.567	.1051	9.514	1.006	.9945	1.4661	84° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	Radians	sin θ	csc θ	tan θ	cot θ	sec θ	cos θ		
6° 00′	.1047	.1045	9.567	.1051	9.514	1.006	.9945	1.4661	84° 00′
10	076	074	9.309	080	9.255	006	942	632	50
20	105	103	9.065	110	9.010	006	939	603	40
30	.1134	.1132	8.834	.1139	8.777	1.006	.9936	1.4573	30
40	164	161	8.614	169	8.556	007	932	544	20
50	193	190	8.405	198	8.345	007	929	515	10
7° 00′	.1222	.1219	8.206	.1228	8.144	1.008	.9925	1.4486	83° 00′
10	251	248	8.016	257	7.953	008	922	457	50
20	280	276	7.834	287	7.770	008	918	428	40
30	.1309	.1305	7.661	.1317	7.596	1.009	.9914	1.4399	30
40	338	334	7.496	346	7.429	009	911	370	20
50	367	363	7.337	376	7.269	009	907	341	10
8° 00′	.1396	.1392	7.185	.1405	7.115	1.010	.9903	1.4312	82° 00′
10	425	421	7.040	435	6.968	010	899	283	50
20	454	449	6.900	465	6.827	011	894	254	40
30	.1484	.1478	6.765	.1495	6.691	1.011	.9890	1.4224	30
40	513	507	6.636	524	6.561	012	886	195	20
50	542	536	6.512	554	6.435	012	881	166	10
9° 00′	.1571	.1564	6.392	.1584	6.314	1.012	.9877	1.4137	81° 00′
10	600	593	277	614	197	013	872	108	50
20	629	622	166	644	084	013	868	079	40
30	.1658	.1650	6.059	.1673	5.976	1.014	.9863	1.4050	30
40	687	679	5.955	703	871	014	858	1.4021	20
50	716	708	855	733	769	015	853	992	10
10° 00′	.1745	.1736	5.759	.1763	5.671	1.015	.9848	1.3963	80° 00′
10	774	765	665	793	576	016	843	934	50
20	804	794	575	823	485	016	838	904	40
30	.1833	.1822	5.487	.1853	5.396	1.017	.9833	1.3875	30
40	862	851	403	883	309	018	827	846	20
50	891	880	320	914	226	018	822	817	10
11° 00′	.1920	.1908	5.241	.1944	5.145	1.019	.9816	1.3788	79° 00′
10	949	937	164	974	066	019	811	759	50
20	978	965	089	.2004	4.989	020	805	730	40
30	.2007	.1994	5.016	.2035	4.915	1.020	.9799	1.3701	30
40	036	.2022	4.945	065	843	021	793	672	20
50	065	051	876	095	773	022	787	643	10
12° 00′	.2094	.2079	4.810	.2126	4.705	1.022	.9781	1.3614	78° 00′
10	123	108	745	156	638	023	775	584	50
20	153	136	682	186	574	024	769	555	40
30	.2182	.2164	4.620	.2217	4.511	1.024	.9763	1.3526	30
40	211	193	560	247	449	025	757	497	20
50	240	221	502	278	390	026	750	468	10
13° 00′	.2269	.2250	4.445	.2309	4.331	1.026	.9744	1.3439	77° 00′
		cos θ	sec θ	cot θ	tan θ	csc θ	sin θ	Radians	Degrees
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
13° 00′	.2269	.2250	4.445	.2309	4.331	1.026	.9744	1.3439	**77° 00′**
10	298	278	390	339	275	027	737	410	50
20	327	306	336	370	219	028	730	381	40
30	.2356	.2334	4.284	.2401	4.165	1.028	.9724	1.3352	30
40	385	363	232	432	113	029	717	323	20
50	414	391	182	462	061	030	710	294	10
14° 00′	.2443	.2419	4.134	.2493	4.011	1.031	.9703	1.3265	**76° 00′**
10	473	447	086	524	3.962	031	696	235	50
20	502	476	039	555	914	032	689	206	40
30	.2531	.2504	3.994	.2586	3.867	1.033	.9681	1.3177	30
40	560	532	950	617	821	034	674	148	20
50	589	560	906	648	776	034	667	119	10
15° 00′	.2618	.2588	3.864	.2679	3.732	1.035	.9659	1.3090	**75° 00′**
10	647	616	822	711	689	036	652	061	50
20	676	644	782	742	647	037	644	032	40
30	.2705	.2672	3.742	.2773	3.606	1.038	.9636	1.3003	30
40	734	700	703	805	566	039	628	974	20
50	763	728	665	836	526	039	621	945	10
16° 00′	.2793	.2756	3.628	.2867	3.487	1.040	.9613	1.2915	**74° 00′**
10	822	784	592	899	450	041	605	886	50
20	851	812	556	931	412	042	596	857	40
30	.2880	.2840	3.521	.2962	3.376	1.043	.9588	1.2828	30
40	909	868	487	994	340	044	580	799	20
50	938	896	453	.3026	305	045	572	770	10
17° 00′	.2967	.2924	3.420	.3057	3.271	1.046	.9563	1.2741	**73° 00′**
10	996	952	388	089	237	047	555	712	50
20	.3025	979	357	121	204	048	546	683	40
30	.3054	.3007	3.326	.3153	3.172	1.048	.9537	1.2654	30
40	083	035	295	185	140	049	528	625	20
50	113	062	265	217	108	050	520	595	10
18° 00′	.3142	.3090	3.236	.3249	3.078	1.051	.9511	1.2566	**72° 00′**
10	171	118	207	281	047	052	502	537	50
20	200	145	179	314	018	053	492	508	40
30	.3229	.3173	3.152	.3346	2.989	1.054	.9483	1.2479	30
40	258	201	124	378	960	056	474	450	20
50	287	228	098	411	932	057	465	421	10
19° 00′	.3316	.3256	3.072	.3443	2.904	1.058	.9455	1.2392	**71° 00′**
10	345	283	046	476	877	059	446	363	50
20	374	311	021	508	850	060	436	334	40
30	.3403	.3338	2.996	.3541	2.824	1.061	.9426	1.2305	30
40	432	365	971	574	798	062	417	275	20
50	462	393	947	607	773	063	407	246	10
20° 00′	.3491	.3420	2.924	.3640	2.747	1.064	.9397	1.2217	**70° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
20° 00′	.3491	.3420	2.924	.3640	2.747	1.064	.9397	1.2217	**70° 00′**
10	520	448	901	673	723	065	387	188	50
20	549	475	878	706	699	066	377	159	40
30	.3578	.3502	2.855	.3739	2.675	1.068	.9367	1.2130	30
40	607	529	833	772	651	069	356	101	20
50	636	557	812	805	628	070	346	072	10
21° 00′	.3665	.3584	2.790	.3839	2.605	1.071	.9336	1.2043	**69° 00′**
10	694	611	769	872	583	072	325	1.2014	50
20	723	638	749	906	560	074	315	985	40
30	.3752	.3665	2.729	.3939	2.539	1.075	.9304	1.1956	30
40	782	692	709	973	517	076	293	926	20
50	811	719	689	.4006	496	077	283	897	10
22° 00′	.3840	.3746	2.669	.4040	2.475	1.079	.9272	1.1868	**68° 00′**
10	869	773	650	074	455	080	261	839	50
20	898	800	632	108	434	081	250	810	40
30	.3927	.3827	2.613	.4142	2.414	1.082	.9239	1.1781	30
40	956	854	595	176	394	084	228	752	20
50	985	881	577	210	375	085	216	723	10
23° 00′	.4014	.3907	2.559	.4245	2.356	1.086	.9205	1.1694	**67° 00′**
10	043	934	542	279	337	088	194	665	50
20	072	961	525	314	318	089	182	636	40
30	.4102	.3987	2.508	.4348	2.300	1.090	.9171	1.1606	30
40	131	.4014	491	383	282	092	159	577	20
50	160	041	475	417	264	093	147	548	10
24° 00′	.4189	.4067	2.459	.4452	2.246	1.095	.9135	1.1519	**66° 00′**
10	218	094	443	487	229	096	124	490	50
20	247	120	427	522	211	097	112	461	40
30	.4276	.4147	2.411	.4557	2.194	1.099	.9100	1.1432	30
40	305	173	396	592	177	100	088	403	20
50	334	200	381	628	161	102	075	374	10
25° 00′	.4363	.4226	2.366	.4663	2.145	1.103	.9063	1.1345	**65° 00′**
10	392	253	352	699	128	105	051	316	50
20	422	279	337	734	112	106	038	286	40
30	.4451	.4305	2.323	.4770	2.097	1.108	.9026	1.1257	30
40	480	331	309	806	081	109	013	228	20
50	509	358	295	841	066	111	001	199	10
26° 00′	.4538	.4384	2.281	.4877	2.050	1.113	.8988	1.1170	**64° 00′**
10	567	410	268	913	035	114	975	141	50
20	596	436	254	950	020	116	962	112	40
30	.4625	.4462	2.241	.4986	2.006	1.117	.8949	1.1083	30
40	654	488	228	.5022	1.991	119	936	054	20
50	683	514	215	059	977	121	923	1.1025	10
27° 00′	.4712	.4540	2.203	.5095	1.963	1.122	.8910	1.0996	**63° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
27° 00	.4712	.4540	2.203	.5095	1.963	1.122	.8910	1.0996	**63° 00′**
10	741	566	190	132	949	124	897	966	50
20	771	592	178	169	935	126	884	937	40
30	.4800	.4617	2.166	.5206	1.921	1.127	.8870	1.0908	30
40	829	643	154	243	907	129	857	879	20
50	858	669	142	280	894	131	843	850	10
28° 00′	.4887	.4695	2.130	.5317	1.881	1.133	.8829	1.0821	**62° 00′**
10	916	720	118	354	868	134	816	792	50
20	945	746	107	392	855	136	802	763	40
30	.4974	.4772	2.096	.5430	1.842	1.138	.8788	1.0734	30
40	.5003	797	085	467	829	140	774	705	20
50	032	823	074	505	816	142	760	676	10
29° 00′	.5061	.4848	2.063	.5543	1.804	1.143	.8746	1.0647	**61° 00′**
10	091	874	052	581	792	145	732	617	50
20	120	899	041	619	780	147	718	588	40
30	.5149	.4924	2.031	.5658	1.767	1.149	.8704	1.0559	30
40	178	950	020	696	756	151	689	530	20
50	207	975	010	735	744	153	675	501	10
30° 00′	.5236	.5000	2.000	.5774	1.732	1.155	.8660	1.0472	**60° 00′**
10	265	025	1.990	812	720	157	646	443	50
20	294	050	980	851	709	159	631	414	40
30	.5323	.5075	1.970	.5890	1.698	1.161	.8616	1.0385	30
40	352	100	961	930	686	163	601	356	20
50	381	125	951	969	675	165	587	327	10
31° 00′	.5411	.5150	1.942	.6009	1.664	1.167	.8572	1.0297	**59° 00′**
10	440	175	932	048	653	169	557	268	50
20	469	200	923	088	643	171	542	239	40
30	.5498	.5225	1.914	.6128	1.632	1.173	.8526	1.0210	30
40	527	250	905	168	621	175	511	181	20
50	556	275	896	208	611	177	496	152	10
32° 00′	.5585	.5299	1.887	.6249	1.600	1.179	.8480	1.0123	**58° 00′**
10	614	324	878	289	590	181	465	094	50
20	643	348	870	330	580	184	450	065	40
30	.5672	.5373	1.861	.6371	1.570	1.186	.8434	1.0036	30
40	701	398	853	412	560	188	418	1.0007	20
50	730	422	844	453	550	190	403	977	10
33° 00′	.5760	.5446	1.836	.6494	1.540	1.192	.8387	.9948	**57° 00′**
10	789	471	828	536	530	195	371	919	50
20	818	495	820	577	520	197	355	890	40
30	.5847	.5519	1.812	.6619	1.511	1.199	.8339	.9861	30
40	876	544	804	661	501	202	323	832	20
50	905	568	796	703	1.492	204	307	803	10
34° 00′	.5934	.5592	1.788	.6745	1.483	1.206	.8290	.9774	**56° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
34° 00′	.5934	.5592	1.788	.6745	1.483	1.206	.8290	.9774	**56° 00′**
10	963	616	781	787	473	209	274	745	50
20	992	640	773	830	464	211	258	716	40
30	.6021	.5664	1.766	.6873	1.455	1.213	.8241	.9687	30
40	050	688	758	916	446	216	225	657	20
50	080	712	751	959	437	218	208	628	10
35° 00′	.6109	.5736	1.743	.7002	1.428	1.221	.8192	.9599	**55° 00′**
10	138	760	736	046	419	223	175	570	50
20	167	783	729	089	411	226	158	541	40
30	.6196	.5807	1.722	.7133	1.402	1.228	.8141	.9512	30
40	225	831	715	177	393	231	124	483	20
50	254	854	708	221	385	233	107	454	10
36° 00′	.6283	.5878	1.701	.7265	1.376	1.236	.8090	.9425	**54° 00′**
10	312	901	695	310	368	239	073	396	50
20	341	925	688	355	360	241	056	367	40
30	.6370	.5948	1.681	.7400	1.351	1.244	.8039	.9338	30
40	400	972	675	445	343	247	021	308	20
50	429	995	668	490	335	249	004	279	10
37° 00′	.6458	.6018	1.662	.7536	1.327	1.252	.7986	.9250	**53° 00′**
10	487	041	655	581	319	255	969	221	50
20	516	065	649	627	311	258	951	192	40
30	.6545	.6088	1.643	.7673	1.303	1.260	.7934	.9163	30
40	574	111	636	720	295	263	916	134	20
50	603	134	630	766	288	266	898	105	10
38° 00′	.6632	.6157	1.624	.7813	1.280	1.269	.7880	.9076	**52° 00′**
10	661	180	618	860	272	272	862	047	50
20	690	202	612	907	265	275	844	.9018	40
30	.6720	.6225	1.606	.7954	1.257	1.278	.7826	.8988	30
40	749	248	601	.8002	250	281	808	959	20
50	778	271	595	050	242	284	790	930	10
39° 00′	.6807	.6293	1.589	.8098	1.235	1.287	.7771	.8901	**51° 00′**
10	836	316	583	146	228	290	753	872	50
20	865	338	578	195	220	293	735	843	40
30	.6894	.6361	1.572	.8243	1.213	1.296	.7716	.8814	30
40	923	383	567	292	206	299	698	785	20
50	952	406	561	342	199	302	679	756	10
40° 00′	.6981	.6428	1.556	.8391	1.192	1.305	.7660	.8727	**50° 00′**
10	.7010	450	550	441	185	309	642	698	50
20	039	472	545	491	178	312	623	668	40
30	.7069	.6494	1.540	.8541	1.171	1.315	.7604	.8639	30
40	098	517	535	591	164	318	585	610	20
50	127	539	529	642	157	322	566	581	10
41° 00′	.7156	.6561	1.524	.8693	1.150	1.325	.7547	.8552	**49° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE I—*continued*

Angle θ									
Degrees	**Radians**	**sin θ**	**csc θ**	**tan θ**	**cot θ**	**sec θ**	**cos θ**		
41° 00′	.7156	.6561	1.524	.8693	1.150	1.325	.7547	.8552	**49° 00′**
10	185	583	519	744	144	328	528	523	50
20	214	604	514	796	137	332	509	494	40
30	.7243	.6626	1.509	.8847	1.130	1.335	.7490	.8465	30
40	272	648	504	899	124	339	470	436	20
50	301	670	499	952	117	342	451	407	10
42° 00′	.7330	.6691	1.494	.9004	1.111	1.346	.7431	.8378	**48° 00′**
10	359	713	490	057	104	349	412	348	50
20	389	734	485	110	098	353	392	319	40
30	.7418	.6756	1.480	.9163	1.091	1.356	.7373	.8290	30
40	447	777	476	.217	085	360	353	261	20
50	476	799	471	271	079	364	333	232	10
43° 00′	.7505	.6820	1.466	.9325	1.072	1.367	.7314	.8203	**47° 00′**
10	534	841	462	380	066	371	294	174	50
20	563	862	457	435	060	375	274	145	40
30	.7592	.6884	1.453	.9490	1.054	1.379	.7254	.8116	30
40	621	905	448	545	048	382	234	087	20
50	650	926	444	601	042	386	214	058	10
44° 00′	.7679	.6947	1.440	.9657	1.036	1.390	.7193	.8029	**46° 00′**
10	709	967	435	713	030	394	173	.7999	50
20	738	988	431	770	024	398	153	970	40
30	.7767	.7009	1.427	.9827	1.018	1.402	.7133	.7941	30
40	796	030	423	884	012	406	112	912	20
50	825	050	418	942	006	410	092	883	10
45° 00′	.7854	.7071	1.414	1.000	1.000	1.414	.7071	.7854	**45° 00′**
		cos θ	**sec θ**	**cot θ**	**tan θ**	**csc θ**	**sin θ**	**Radians**	**Degrees**
								Angle θ	

TABLE II

Four-Place Values of Trigonometric Functions
Real Numbers *u*, or Angles θ, in Radians and Degrees

Real Number *u* or θ radians	θ degrees	sin *u* or sin θ	csc *u* or csc θ	tan *u* or tan θ	cot *u* or cot θ	sec *u* or sec θ	cos *u* or cos θ
0.00	0° 00′	0.0000	No value	0.0000	No value	1.000	1.000
.01	0° 34′	.0100	100.0	.0100	100.0	1.000	1.000
.02	1° 09′	.0200	50.00	.0200	49.99	1.000	0.9998
.03	1° 43′	.0300	33.34	.0300	33.32	1.000	0.9996
.04	2° 18′	.0400	25.01	.0400	24.99	1.001	0.9992
0.05	2° 52′	0.0500	20.01	0.0500	19.98	1.001	0.9988
.06	3° 26′	.0600	16.68	.0601	16.65	1.002	.9982
.07	4° 01′	.0699	14.30	.0701	14.26	1.002	.9976
.08	4° 35′	.0799	12.51	.0802	12.47	1.003	.9968
.09	5° 09′	.0899	11.13	.0902	11.08	1.004	.9960
0.10	5° 44′	0.0998	10.02	0.1003	9.967	1.005	0.9950
.11	6° 18′	.1098	9.109	.1104	9.054	1.006	.9940
.12	6° 53′	.1197	8.353	.1206	8.293	1.007	.9928
.13	7° 27′	.1296	7.714	.1307	7.649	1.009	.9916
.14	8° 01′	.1395	7.166	.1409	7.096	1.010	.9902
0.15	8° 36′	0.1494	6.692	0.1511	6.617	1.011	0.9888
.16	9° 10′	.1593	6.277	.1614	6.197	1.013	.9872
.17	9° 44′	.1692	5.911	.1717	5.826	1.015	.9856
.18	10° 19′	.1790	5.586	.1820	5.495	1.016	.9838
.19	10° 53′	.1889	5.295	.1923	5.200	1.018	.9820
0.20	11° 28′	0.1987	5.033	0.2027	4.933	1.020	0.9801
.21	12° 02′	.2085	4.797	.2131	4.692	1.022	.9780
.22	12° 36′	.2182	4.582	.2236	4.472	1.025	.9759
.23	13° 11′	.2280	4.386	.2341	4.271	1.027	.9737
.24	13° 45′	.2377	4.207	.2447	4.086	1.030	.9713
0.25	14° 19′	0.2474	4.042	0.2553	3.916	1.032	0.9689
.26	14° 54′	.2571	3.890	.2660	3.759	1.035	.9664
.27	15° 28′	.2667	3.749	.2768	3.613	1.038	.9638
.28	16° 03′	.2764	3.619	.2876	3.478	1.041	.9611
.29	16° 37′	.2860	3.497	.2984	3.351	1.044	.9582
0.30	17° 11′	0.2955	3.384	0.3093	3.233	1.047	0.9553
.31	17° 46′	.3051	3.278	.3203	3.122	1.050	.9523
.32	18° 20′	.3146	3.179	.3314	3.018	1.053	.9492
.33	18° 54′	.3240	3.086	.3425	2.920	1.057	.9460
.34	19° 29′	.3335	2.999	.3537	2.827	1.061	.9428
0.35	20° 03′	0.3429	2.916	0.3650	2.740	1.065	0.9394
Real Number *u* or θ radians	θ degrees	sin *u* or sin θ	csc *u* or csc θ	tan *u* or tan θ	cot *u* or cot θ	sec *u* or sec θ	cos *u* or cos θ

TABLE II—*continued*

Real Number *u* or θ radians	θ degrees	sin *u* or sin θ	csc *u* or csc θ	tan *u* or tan θ	cot *u* or cot θ	sec *u* or sec θ	cos *u* or cos θ
0.35	20° 03′	0.3429	2.916	0.3650	2.740	1.065	0.9394
.36	20° 38′	.3523	2.839	.3764	2.657	1.068	.9359
.37	21° 12′	.3616	2.765	.3879	2.578	1.073	.9323
.38	21° 46′	.3709	2.696	.3994	2.504	1.077	.9287
.39	22° 21′	.3802	2.630	.4111	2.433	1.081	.9249
0.40	22° 55′	0.3894	2.568	0.4228	2.365	1.086	0.9211
.41	23° 29′	.3986	2.509	.4346	2.301	1.090	.9171
.42	24° 04′	.4078	2.452	.4466	2.239	1.095	.9131
.43	24° 38′	.4169	2.399	.4586	2.180	1.100	.9090
.44	25° 13′	.4259	2.348	.4708	2.124	1.105	.9048
0.45	25° 47′	0.4350	2.299	0.4831	2.070	1.111	0.9004
.46	26° 21′	.4439	2.253	.4954	2.018	1.116	.8961
.47	26° 56′	.4529	2.208	.5080	1.969	1.122	.8916
.48	27° 30′	.4618	2.166	.5206	1.921	1.127	.8870
.49	28° 04′	.4706	2.125	.5334	1.875	1.133	.8823
0.50	28° 39′	0.4794	2.086	0.5463	1.830	1.139	0.8776
.51	29° 13′	.4882	2.048	.5594	1.788	1.146	.8727
.52	29° 48′	.4969	2.013	.5726	1.747	1.152	.8678
.53	30° 22′	.5055	1.978	.5859	1.707	1.159	.8628
.54	30° 56′	.5141	1.945	.5994	1.668	1.166	.8577
0.55	31° 31′	0.5227	1.913	0.6131	1.631	1.173	0.8525
.56	32° 05′	.5312	1.883	.6269	1.595	1.180	.8473
.57	32° 40′	.5396	1.853	.6410	1.560	1.188	.8419
.58	33° 14′	.5480	1.825	.6552	1.526	1.196	.8365
.59	33° 48′	.5564	1.797	.6696	1.494	1.203	.8309
0.60	34° 23′	0.5646	1.771	0.6841	1.462	1.212	0.8253
.61	34° 57′	.5729	1.746	.6989	1.431	1.220	.8196
.62	35° 31′	.5810	1.721	.7139	1.401	1.229	.8139
.63	36° 06′	.5891	1.697	.7291	1.372	1.238	.8080
.64	36° 40′	.5972	1.674	.7445	1.343	1.247	.8021
0.65	37° 15′	0.6052	1.652	0.7602	1.315	1.256	0.7961
.66	37° 49′	.6131	1.631	.7761	1.288	1.266	.7900
.67	38° 23′	.6210	1.610	.7923	1.262	1.276	.7838
.68	38° 58′	.6288	1.590	.8087	1.237	1.286	.7776
.69	39° 32′	.6365	1.571	.8253	1.212	1.297	.7712
0.70	40° 06′	0.6442	1.552	0.8423	1.187	1.307	0.7648
Real Number *u* or θ radians	**θ degrees**	**sin *u* or sin θ**	**csc *u* or csc θ**	**tan *u* or tan θ**	**cot *u* or cot θ**	**sec *u* or sec θ**	**cos *u* or cos θ**

TABLE II—*continued*

Real Number *u* or θ radians	θ degrees	sin *u* or sin θ	csc *u* or csc θ	tan *u* or tan θ	cot *u* or cot θ	sec *u* or sec θ	cos *u* or cos θ
0.70	40° 06′	0.6442	1.552	0.8423	1.187	1.307	0.7648
.71	40° 41′	.6518	1.534	.8595	1.163	1.319	.7584
.72	41° 15′	.6594	1.517	.8771	1.140	1.330	.7518
.73	41° 50′	.6669	1.500	.8949	1.117	1.342	.7452
.74	42° 24′	.6743	1.483	.9131	1.095	1.354	.7385
0.75	42° 58′	0.6816	1.467	0.9316	1.073	1.367	0.7317
.76	43° 33′	.6889	1.452	.9505	1.052	1.380	.7248
.77	44° 07′	.6961	1.436	.9697	1.031	1.393	.7179
.78	44° 41′	.7033	1.422	.9893	1.011	1.407	.7109
.79	45° 16′	.7104	1.408	1.009	.9908	1.421	.7038
0.80	45° 50′	0.7174	1.394	1.030	0.9712	1.435	0.6967
.81	46° 25′	.7243	1.381	1.050	.9520	1.450	.6895
.82	46° 59′	.7311	1.368	1.072	.9331	1.466	.6822
.83	47° 33′	.7379	1.355	1.093	.9146	1.482	.6749
.84	48° 08′	.7446	1.343	1.116	.8964	1.498	.6675
0.85	48° 42′	0.7513	1.331	1.138	0.8785	1.515	0.6600
.86	49° 16′	.7578	1.320	1.162	.8609	1.533	.6524
.87	49° 51′	.7643	1.308	1.185	.8437	1.551	.6448
.88	50° 25′	.7707	1.297	1.210	.8267	1.569	.6372
.89	51° 00′	.7771	1.287	1.235	.8100	1.589	.6294
0.90	51° 34′	0.7833	1.277	1.260	0.7936	1.609	0.6216
.91	52° 08′	.7895	1.267	1.286	.7774	1.629	.6137
.92	52° 43′	.7956	1.257	1.313	.7615	1.651	.6058
.93	53° 17′	.8016	1.247	1.341	.7458	1.673	.5978
.94	53° 51′	.8076	1.238	1.369	.7303	1.696	.5898
0.95	54° 26′	0.8134	1.229	1.398	0.7151	1.719	0.5817
.96	55° 00′	.8192	1.221	1.428	.7001	1.744	.5735
.97	55° 35′	.8249	1.212	1.459	.6853	1.769	.5653
.98	56° 09′	.8305	1.204	1.491	.6707	1.795	.5570
.99	56° 43′	.8360	1.196	1.524	.6563	1.823	.5487
1.00	57° 18′	0.8415	1.188	1.557	0.6421	1.851	0.5403
1.01	57° 52′	.8468	1.181	1.592	.6281	1.880	.5319
1.02	58° 27′	.8521	1.174	1.628	.6142	1.911	.5234
1.03	59° 01′	.8573	1.166	1.665	.6005	1.942	.5148
1.04	59° 35′	.8624	1.160	1.704	.5870	1.975	.5062
1.05	60° 10′	0.8674	1.153	1.743	0.5736	2.010	0.4976
Real Number *u* or θ radians	**θ degrees**	**sin *u* or sin θ**	**csc *u* or csc θ**	**tan *u* or tan θ**	**cot *u* or cot θ**	**sec *u* or sec θ**	**cos *u* or cos θ**

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
1.05	60° 10′	0.8674	1.153	1.743	0.5736	2.010	0.4976
1.06	60° 44′	8724	1.146	1.784	.5604	2.046	.4889
1.07	61° 18′	.8772	1.140	1.827	.5473	2.083	.4801
1.08	61° 53′	.8820	1.134	1.871	.5344	2.122	.4713
1.09	62° 27′	.8866	1.128	1.917	.5216	2.162	.4625
1.10	63° 02′	0.8912	1.122	1.965	0.5090	2.205	0.4536
1.11	63° 36′	.8957	1.116	2.014	.4964	2.249	.4447
1.12	64° 10′	.9001	1.111	2.066	.4840	2.295	.4357
1.13	64° 45′	.9044	1.106	2.120	.4718	2.344	.4267
1.14	65° 19′	.9086	1.101	2.176	.4596	2.395	.4176
1.15	65° 53′	0.9128	1.096	2.234	0.4475	2.448	0.4085
1.16	66° 28′	.9168	1.091	2.296	.4356	2.504	.3993
1.17	67° 02′	.9208	1.086	2.360	.4237	2.563	.3902
1.18	67° 37′	.9246	1.082	2.247	.4120	2.625	.3809
1.19	68° 11′	.9284	1.077	2.498	.4003	2.691	.3717
1.20	68° 45′	0.9320	1.073	2.572	0.3888	2.760	0.3624
1.21	69° 20′	.9356	1.069	2.650	.3773	2.833	.3530
1.22	69° 54′	.9391	1.065	2.733	.3659	2.910	.3436
1.23	70° 28′	.9425	1.061	2.820	.3546	2.992	.3342
1.24	71° 03′	.9458	1.057	2.912	.3434	3.079	.3248
1.25	71° 37′	0.9490	1.054	3.010	0.3323	3.171	0.3153
1.26	72° 12′	.9521	1.050	3.113	.3212	3.270	.3058
1.27	72° 46′	.9551	1.047	3.224	.3102	3.375	.2963
1.28	73° 20′	.9580	1.044	3.341	.2993	3.488	.2867
1.29	73° 55′	.9608	1.041	3.467	.2884	3.609	.2771
1.30	74° 29′	0.9636	1.038	3.602	0.2776	3.738	0.2675
1.31	75° 03′	.9662	1.035	3.747	.2669	3.878	.2579
1.32	75° 38′	.9687	1.032	3.903	.2562	4.029	.2482
1.33	76° 12′	.9711	1.030	4.072	.2456	4.193	.2385
1.34	76° 47′	.9735	1.027	4.256	.2350	4.372	.2288
1.35	77° 21′	0.9757	1.025	4.455	0.2245	4.566	0.2190
1.36	77° 55′	.9779	1.023	4.673	.2140	4.779	.2092
1.37	78° 30′	.9799	1.021	4.913	.2035	5.014	.1994
1.38	79° 04′	.9819	1.018	5.177	.1931	5.273	.1896
1.39	79° 38′	.9837	1.017	5.471	.1828	5.561	.1798
1.40	80° 13′	0.9854	1.015	5.798	0.1725	5.883	0.1700
Real Number u or θ radians	**θ degrees**	**sin u or sin θ**	**csc u or csc θ**	**tan u or tan θ**	**cot u or cot θ**	**sec u or sec θ**	**cos u or cos θ**

TABLE II—*continued*

Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ
1.40	80° 13′	0.9854	1.015	5.798	0.1725	5.883	0.1700
1.41	80° 47′	.9871	1.013	6.165	.1622	6.246	.1601
1.42	81° 22′	.9887	1.011	6.581	.1519	6.657	.1502
1.43	81° 56′	.9901	1.010	7.055	.1417	7.126	.1403
1.44	82° 30′	.9915	1.009	7.602	.1315	7.667	.1304
1.45	83° 05′	0.9927	1.007	8.238	0.1214	8.299	0.1205
1.46	83° 39′	.9939	1.006	8.989	.1113	9.044	.1106
1.47	84° 13′	.9949	1.005	9.887	.1011	9.938	.1006
1.48	84° 48′	.9959	1.004	10.98	.0910	11.03	.0907
1.49	85° 22′	.9967	1.003	12.35	.0810	12.39	.0807
1.50	85° 57′	0.9975	1.003	14.10	0.0709	14.14	0.0707
1.51	86° 31′	.9982	1.002	16.43	.0609	16.46	.0608
1.52	87° 05′	.9987	1.001	19.67	.0508	19.69	.0508
1.53	87° 40′	.9992	1.001	24.50	.0408	24.52	.0408
1.54	88° 14′	.9995	1.000	32.46	.0308	32.48	.0308
1.55	88° 49′	0.9998	1.000	48.08	0.0208	48.09	0.0208
1.56	89° 23′	.9999	1.000	92.62	.0108	92.63	.0108
1.57	89° 57′	1.000	1.000	1256	.0008	1256	.0008
Real Number u or θ radians	θ degrees	sin u or sin θ	csc u or csc θ	tan u or tan θ	cot u or cot θ	sec u or sec θ	cos u or cos θ

TABLE III

Four-Place Logarithms of Numbers from 1 to 10

To extend the table write the number N as

$N = n \times 10^c$, $1 \leqq n < 10$, c an integer, and use

$\log N = \log n + c$.

n	0	1	2	3	4	5	6	7	8	9
1.0	+0.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
1.1	.0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
1.2	.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
1.3	.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
1.4	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
1.5	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
1.6	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
1.7	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
1.8	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
1.9	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
2.0	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
2.1	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
2.2	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
2.3	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
2.4	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
2.5	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
2.6	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
2.7	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
2.8	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
2.9	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
3.0	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
3.1	.4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
3.2	.5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
3.3	.5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
3.4	.5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
3.5	.5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
3.6	.5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
3.7	.5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
3.8	.5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
3.9	.5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
4.0	.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
4.1	.6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
4.2	.6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
4.3	.6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
4.4	.6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
4.5	.6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
4.6	.6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
4.7	.6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
4.8	.6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
4.9	.6902	6911	6920	6928	6937	6946	6955	6964	6972	6981

TABLE III—*continued*

n	0	1	2	3	4	5	6	7	8	9
5.0	+.6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
5.1	.7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
5.2	.7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
5.3	.7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
5.4	.7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
5.5	.7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
5.6	.7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
5.7	.7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
5.8	.7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
5.9	.7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
6.0	.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
6.1	.7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
6.2	.7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
6.3	.7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
6.4	.8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
6.5	.8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
6.6	.8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
6.7	.8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
6.8	.8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
6.9	.8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
7.0	.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
7.1	.8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
7.2	.8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
7.3	.8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
7.4	.8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
7.5	.8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
7.6	.8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
7.7	.8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
7.8	.8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
7.9	.8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
8.0	.9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
8.1	.9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
8.2	.9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
8.3	.9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
8.4	.9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
8.5	.9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
8.6	.9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
8.7	.9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
8.8	.9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
8.9	.9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
9.0	.9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
9.1	.9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
9.2	.9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
9.3	.9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
9.4	.9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
9.5	.9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
9.6	.9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
9.7	.9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
9.8	.9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
9.9	.9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

TABLE IV

Four-Place Logarithms of Trigonometric Functions
Angle θ in Degrees

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
0° 00′	No value	No value	No value	No value	10.0000	10.0000	**90° 00′**
10′	7.4637	12.5363	7.4637	12.5363	.0000	.0000	50′
20′	.7648	.2352	.7648	.2352	.0000	.0000	40′
30′	7.9408	12.0592	7.9409	12.0591	.0000	.0000	30′
40′	8.0658	11.9342	8.0658	11.9342	.0000	.0000	20′
50′	.1627	.8373	.1627	.8373	.0000	10.0000	10′
1° 00′	8.2419	11.7581	8.2419	11.7581	10.0001	9.9999	**89° 00′**
10′	.3088	.6912	.3089	.6911	.0001	.9999	50′
20′	.3668	.6332	.3669	.6331	.0001	.9999	40′
30′	.4179	.5821	.4181	.5819	.0001	.9999	30′
40′	.4637	.5363	.4638	.5362	.0002	.9998	20′
50′	.5050	.4950	.5053	.4947	.0002	.9998	10′
2° 00′	8.5428	11.4572	8.5431	11.4569	10.0003	9.9997	**88° 00′**
10′	.5776	.4224	.5779	.4221	.0003	.9997	50′
20′	.6097	.3903	.6101	.3899	.0004	.9996	40′
30′	.6397	.3603	.6401	.3599	.0004	.9996	30′
40′	.6677	.3323	.6682	.3318	.0005	.9995	20′
50′	.6940	.3060	.6945	.3055	.0005	.9995	10′
3° 00′	8.7188	11.2812	8.7194	11.2806	10.0006	9.9994	**87° 00′**
10′	.7423	.2577	.7429	.2571	.0007	.9993	50′
20′	.7645	.2355	.7652	.2348	.0007	.9993	40′
30′	.7857	.2143	.7865	.2135	.0008	.9992	30′
40′	.8059	.1941	.8067	.1933	.0009	.9991	20′
50′	.8251	.1749	.8261	.1739	.0010	.9990	10′
4° 00′	8.8436	11.1564	8.8446	11.1554	10.0011	9.9989	**86° 00′**
10′	.8613	.1387	.8624	.1376	.0011	.9989	50′
20′	.8783	.1217	.8795	.1205	.0012	.9988	40′
30′	.8946	.1054	.8960	.1040	.0013	.9987	30′
40′	.9104	.0896	.9118	.0882	.0014	.9986	20′
50′	.9256	.0744	.9272	.0728	.0015	.9985	10′
5° 00′	8.9403	11.0597	8.9420	11.0580	10.0017	9.9983	**85° 00′**
10′	.9545	.0455	.9563	.0437	.0018	.9982	50′
20′	.9682	.0318	.9701	.0299	.0019	.9981	40′
30′	.9816	.0184	.9836	.0164	.0020	.9980	30′
40′	8.9945	11.0055	8.9966	11.0034	.0021	.9979	20′
50′	9.0070	10.9930	9.0093	10.9907	.0023	.9977	10′
6° 00′	9.0192	10.9808	9.0216	10.9784	10.0024	9.9976	**84° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
6° 00′	9.0192	10.9808	9.0216	10.9784	10.0024	9.9976	**84° 00′**
10′	.0311	.9689	.0336	.9664	.0025	.9975	50′
20′	.0426	.9574	.0453	.9547	.0027	.9973	40′
30′	.0539	.9461	.0567	.9433	.0028	.9972	30′
40′	.0648	.9352	.0678	.9322	.0029	.9971	20′
50′	.0755	.9245	.0786	.9214	.0031	.9969	10′
7° 00′	9.0859	10.9141	9.0891	10.9109	10.0032	9.9968	**83° 00′**
10′	.0961	.9039	.0995	.9005	.0034	.9966	50′
20′	.1060	.8940	.1096	.8904	.0036	.9964	40′
30′	.1157	.8843	.1194	.8806	.0037	.9963	30′
40′	.1252	.8748	.1291	.8709	.0039	.9961	20′
50′	.1345	.8655	.1385	.8615	.0041	.9959	10′
8° 00′	9.1436	10.8564	9.1478	10.8522	10.0042	9.9958	**82° 00′**
10′	.1525	.8475	.1569	.8431	.0044	.9956	50′
20′	.1612	.8388	.1658	.8342	.0046	.9954	40′
30′	.1697	.8303	.1745	.8255	.0048	.9952	30′
40′	.1781	.8219	.1831	.8169	.0050	.9950	20′
50′	.1863	.8137	.1915	.8085	.0052	.9948	10′
9° 00′	9.1943	10.8057	9.1997	10.8003	10.0054	9.9946	**81° 00′**
10′	.2022	.7978	.2078	.7922	.0056	.9944	50′
20′	.2100	.7900	.2158	.7842	.0058	.9942	40′
30′	.2176	.7824	.2236	.7764	.0060	.9940	30′
40′	.2251	.7749	.2313	.7687	.0062	.9938	20′
50′	.2324	.7676	.2389	.7611	.0064	.9936	10′
10° 00′	9.2397	10.7603	9.2463	10.7537	10.0066	9.9934	**80° 00′**
10′	.2468	.7532	.2536	.7464	.0069	.9931	50′
20′	.2538	.7462	.2609	.7391	.0071	.9929	40′
30′	.2606	.7394	.2680	.7320	.0073	.9927	30′
40′	.2674	.7326	.2750	.7250	.0076	.9924	20′
50′	.2740	.7260	.2819	.7181	.0078	.9922	10′
11° 00′	9.2806	10.7194	9.2887	10.7113	10.0081	9.9919	**79° 00′**
10′	.2870	.7130	.2953	.7047	.0083	.9917	50′
20′	.2934	.7066	.3020	.6980	.0086	.9914	40′
30′	.2997	.7003	.3085	.6915	.0088	.9912	30′
40′	.3058	.6942	.3149	.6851	.0091	.9909	20′
50′	.3119	.6881	.3212	.6788	.0093	.9907	10′
12° 00′	9.3179	10.6821	9.3275	10.6725	10.0096	9.9904	**78° 00′**
10′	.3238	.6762	.3336	.6664	.0099	.9901	50′
20′	.3296	.6704	.3397	.6603	.0101	.9899	40′
30′	.3353	.6647	.3458	.6542	.0104	.9896	30′
40′	.3410	.6590	.3517	.6483	.0107	.9893	20′
50′	.3466	.6534	.3576	.6424	.0110	.9890	10′
13° 00′	9.3521	10.6479	9.3634	10.6366	10.0113	9.9887	**77° 00′**
	L cos θ	**L sec θ**	**L cot θ**	**L tan θ**	**L csc θ**	**L sin θ**	**Angle θ**

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
13° 00′	9.3521	10.6479	9.3634	10.6366	10.0113	9.9887	77° 00′
10′	.3575	.6425	.3691	.6309	.0116	.9884	50′
20′	.3629	.6371	.3748	.6252	.0119	.9881	40′
30′	.3682	.6318	.3804	.6196	.0122	.9878	30′
40′	.3734	.6266	.3859	.6141	.0125	.9875	20′
50′	.3786	.6214	.3914	.6086	.0128	.9872	10′
14° 00′	9.3837	10.6163	9.3968	10.6032	10.0131	9.9869	76° 00′
10′	.3887	.6113	.4021	.5979	.0134	.9866	50′
20′	.3937	.6063	.4074	.5926	.0137	.9863	40′
30′	.3986	.6014	.4127	.5873	.0141	.9859	30′
40′	.4035	.5965	.4178	.5822	.0144	.9856	20′
50′	.4083	.5917	.4230	.5770	.0147	.9853	10′
15° 00′	9.4130	10.5870	9.4281	10.5719	10.0151	9.9849	75° 00′
10′	.4177	.5823	.4331	.5669	.0154	.9846	50′
20′	.4223	.5777	.4381	.5619	.0157	.9843	40′
30′	.4269	.5731	.4430	.5570	.0161	.9839	30′
40′	.4314	.5686	.4479	.5521	.0164	.9836	20′
50′	.4359	.5641	.4527	.5473	.0168	.9832	10′
16° 00′	9.4403	10.5597	9.4575	10.5425	10.0172	9.9828	74° 00′
10′	.4447	.5553	.4622	.5378	.0175	.9825	50′
20′	.4491	.5509	.4669	.5331	.0179	.9821	40′
30′	.4533	.5467	.4716	.5284	.0183	.9817	30′
40′	.4576	.5424	.4762	.5238	.0186	.9814	20′
50′	.4618	.5382	.4808	.5192	.0190	.9810	10′
17° 00′	9.4659	10.5341	9.4853	10.5147	10.0194	9.9806	73° 00′
10′	.4700	.5300	.4898	.5102	.0198	.9802	50′
20′	.4741	.5259	.4943	.5057	.0202	.9798	40′
30′	.4781	.5219	.4987	.5013	.0206	.9794	30′
40′	.4821	.5179	.5031	.4969	.0210	.9790	20′
50′	.4861	.5139	.5075	.4925	.0214	.9786	10′
18° 00′	9.4900	10.5100	9.5118	10.4882	10.0218	9.9782	72° 00′
10′	.4939	.5061	.5161	.4839	.0222	.9778	50′
20′	.4977	.5023	.5203	.4797	.0226	.9774	40′
30′	.5015	.4985	.5245	.4755	.0230	.9770	30′
40′	.5052	.4948	.5287	.4713	.0235	.9765	20′
50′	.5090	.4910	.5329	.4671	.0239	.9761	10′
19° 00′	9.5126	10.4874	9.5370	10.4630	10.0243	9.9757	71° 00′
10′	.5163	.4837	.5411	.4589	.0248	.9752	50′
20′	.5199	.4801	.5451	.4549	.0252	.9748	40′
30′	.5235	.4765	.5491	.4509	.0257	.9743	30′
40′	.5270	.4730	.5531	.4469	.0261	.9739	20′
50′	.5306	.4694	.5571	.4429	.0266	.9734	10′
20° 00′	9.5341	10.4659	9.5611	10.4389	10.0270	9.9730	70° 00′
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach −10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
20° 00′	9.5341	10.4659	9.5611	10.4389	10.0270	9.9730	**70° 00′**
10′	.5375	.4625	.5650	.4350	.0275	.9725	50′
20′	.5409	.4591	.5689	.4311	.0279	.9721	40′
30′	.5443	.4557	.5727	.4273	.0284	.9716	30′
40′	.5477	.4523	.5766	.4234	.0289	.9711	20′
50′	.5510	.4490	.5804	.4196	.0294	.9706	10′
21° 00′	9.5543	10.4457	9.5842	10.4158	10.0298	9.9702	**69° 00′**
10′	.5576	.4424	.5879	.4121	.0303	.9797	50′
20′	.5609	.4391	.5917	.4083	.0308	.9692	40′
30′	.5641	.4359	.5954	.4046	.0313	.9687	30′
40′	.5673	.4327	.5991	.4009	.0318	.9682	20′
50′	.5704	.4296	.6028	.3972	.0323	.9677	10′
22° 00′	9.5736	10.4264	9.6064	10.3936	10.0328	9.9672	**68° 00′**
10′	.5767	.4233	.6100	.3900	.0333	.9667	50′
20′	.5798	.4202	.6136	.3864	.0339	.9661	40′
30′	.5828	.4172	.6172	.3828	.0344	.9656	30′
40′	.5859	.4141	.6208	.3792	.0349	.9651	20′
50′	.5889	.4111	.6243	.3757	.0354	.9646	10′
23° 00′	9.5919	10.4081	9.6279	10.3721	10.0360	9.9640	**67° 00′**
10′	.5948	.4052	.6314	.3686	.0365	.9635	50′
20′	.5978	.4022	.6348	.3652	.0371	.9629	40′
30′	.6007	.3993	.6383	.3617	.0376	.9624	30′
40′	.6036	.3964	.6417	.3583	.0382	.9618	20′
50′	.6065	.3935	.6452	.3548	.0387	.9613	10′
24° 00′	9.6093	10.3907	9.6486	10.3514	10.0393	9.9607	**66° 00′**
10′	.6121	.3879	.6520	.3480	.0398	.9602	50′
20′	.6149	.3851	.6553	.3447	.0404	.9596	40′
30′	.6177	.3823	.6587	.3413	.0410	.9590	30′
40′	.6205	.3795	.6620	.3380	.0416	.9584	20′
50′	.6232	.3768	.6654	.3346	.0421	.9579	10′
25° 00′	9.6259	10.3741	9.6687	10.3313	10.0427	9.9573	**65° 00′**
10′	.6286	.3714	.6720	.3280	.0433	.9567	50′
20′	.6313	.3687	.6752	.3248	.0439	.9561	40′
30′	.6340	.3660	.6785	.3215	.0445	.9555	30′
40′	.6366	.3634	.6817	.3183	.0451	.9549	20′
50′	.6392	.3608	.6850	.3150	.0457	.9543	10′
26° 00′	9.6418	10.3582	9.6882	10.3118	10.0463	9.9537	**64° 00′**
10′	.6444	.3556	.6914	.3086	.0470	.9530	50′
20′	.6470	.3530	.6946	.3054	.0476	.9524	40′
30′	.6495	.3505	.6977	.3023	.0482	.9518	30′
40′	.6521	.3479	.7009	.2991	.0488	.9512	20′
50′	.6546	.3454	.7040	.2960	.0495	.9505	10′
27° 00′	9.6570	10.3430	9.7072	10.2928	10.0501	9.9499	**63° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
27° 00′	9.6570	10.3430	9.7072	10.2928	10.0501	9.9499	**63° 00′**
10′	.6595	.3405	.7103	.2897	.0508	.9492	50′
20′	.6620	.3380	.7134	.2866	.0514	.9486	40′
30′	.6644	.3356	.7165	.2835	.0521	.9479	30′
40′	.6668	.3332	.7196	.2804	.0527	.9473	20′
50′	.6692	.3308	.7226	.2774	.0534	.9466	10′
28° 00′	9.6716	10.3284	9.7257	10.2743	10.0541	9.9459	**62° 00′**
10′	.6740	.3260	.7287	.2713	.0547	.9453	50′
20′	.6763	.3237	.7317	.2683	.0554	.9446	40′
30′	.6787	.3213	.7348	.2652	.0561	.9439	30′
40′	.6810	.3190	.7378	.2622	.0568	.9432	20′
50′	.6833	.3167	.7408	.2592	.0575	.9425	10′
29° 00′	9.6856	10.3144	9.7438	10.2562	10.0582	9.9418	**61° 00′**
10′	.6878	.3122	.7467	.2533	.0589	.9411	50′
20′	.6901	.3099	.7497	.2503	.0596	.9404	40′
30′	.6923	.3077	.7526	.2474	.0603	.9397	30′
40′	.6946	.3054	.7556	.2444	.0610	.9390	20′
50′	.6968	.3032	.7585	.2415	.0617	.9383	10′
30° 00′	9.6990	10.3010	9.7614	10.2386	10.0625	9.9375	**60° 00′**
10′	.7012	.2988	.7644	.2356	.0632	.9368	50′
20′	.7033	.2967	.7673	.2327	.0639	.9361	40′
30′	.7055	.2945	.7701	.2299	.0647	.9353	30′
40′	.7076	.2924	.7730	.2270	.0654	.9346	20′
50′	.7097	.2903	.7759	.2241	.0662	.9338	10′
31° 00′	9.7118	10.2882	9.7788	10.2212	10.0669	9.9331	**59° 00′**
10′	.7139	.2861	.7816	.2184	.0677	.9323	50′
20′	.7160	.2840	.7845	.2155	.0685	.9315	40′
30′	.7181	.2819	.7873	.2127	.0692	.9308	30′
40′	.7201	.2799	.7902	.2098	.0700	.9300	20′
50′	.7222	.2778	.7930	.2070	.0708	.9292	10′
32° 00′	9.7242	10.2758	9.7958	10.2042	10.0716	9.9284	**58° 00′**
10′	.7262	.2738	.7986	.2014	.0724	.9276	50′
20′	.7282	.2718	.8014	.1986	.0732	.9268	40′
30′	.7302	.2698	.8042	.1958	.0740	.9260	30′
40′	.7322	.2678	.8070	.1930	.0748	.9252	20′
50′	.7342	.2658	.8097	.1903	.0756	.9244	10′
33° 00′	9.7361	10.2639	9.8125	10.1875	10.0764	9.9236	**57° 00′**
10′	.7380	.2620	.8153	.1847	.0772	.9228	50′
20′	.7400	.2600	.8180	.1820	.0781	.9219	40′
30′	.7419	.2581	.8208	.1792	.0789	.9211	30′
40′	.7438	.2562	.8235	.1765	.0797	.9203	20′
50′	.7457	.2543	.8263	.1737	.0806	.9194	10′
34° 00′	9.7476	10.2524	9.8290	10.1710	10.0814	9.9186	**56° 00′**
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
34° 00′	9.7476	10.2524	9.8290	10.1710	10.0814	9.9186	56° 00′
10′	.7494	.2506	.8317	.1683	.0823	.9177	50′
20′	.7513	.2487	.8344	.1656	.0831	.9169	40′
30′	.7531	.2469	.8371	.1629	.0840	.9160	30′
40′	.7550	.2450	.8398	.1602	.0849	.9151	20′
50′	.7568	.2432	.8425	.1575	.0858	.9142	10′
35° 00′	9.7586	10.2414	9.8452	10.1548	10.0866	9.9134	55° 00′
10′	.7604	.2396	.8479	.1521	.0875	.9125	50′
20′	.7622	.2378	.8506	.1494	.0884	.9116	40′
30′	.7640	.2360	.8533	.1467	.0893	.9107	30′
40′	.7657	.2343	.8559	.1441	.0902	.9098	20′
50′	.7675	.2325	.8586	.1414	.0911	.9089	10′
36° 00′	9.7692	10.2308	9.8613	10.1387	10.0920	9.9080	54° 00′
10′	.7710	.2290	.8639	.1361	.0930	.9070	50′
20′	.7727	.2273	.8666	.1334	.0939	.9061	40′
30′	.7744	.2256	.8692	.1308	.0948	.9052	30′
40′	.7761	.2239	.8718	.1282	.0958	.9042	20′
50′	.7778	.2222	.8745	.1255	.0967	.9033	10′
37° 00′	9.7795	10.2205	9.8771	10.1229	10.0977	9.9023	53° 00′
10′	.7811	.2189	.8797	.1203	.0986	.9014	50′
20′	.7828	.2172	.8824	.1176	.0996	.9004	40′
30′	.7844	.2156	.8850	.1150	.1005	.8995	30′
40′	.7861	.2139	.8876	.1124	.1015	.8985	20′
50′	.7877	.2123	.8902	.1098	.1025	.8975	10′
38° 00′	9.7893	10.2107	9.8928	10.1072	10.1035	9.8965	52° 00′
10′	.7910	.2090	.8954	.1046	.1045	.8955	50′
20′	.7926	.2074	.8980	.1020	.1055	.8945	40′
30′	.7941	.2059	.9006	.0994	.1065	.8935	30′
40′	.7957	.2043	.9032	.0968	.1075	.8925	20′
50′	.7973	.2027	.9058	.0942	.1085	.8915	10′
39° 00′	9.7989	10.2011	9.9084	10.0916	10.1095	9.8905	51° 00′
10′	.8004	.1996	.9110	.0890	.1105	.8895	50′
20′	.8020	.1980	.9135	.0865	.1116	.8884	40′
30′	.8035	.1965	.9161	.0839	.1126	.8874	30′
40′	.8050	.1950	.9187	.0813	.1136	.8864	20′
50′	.8066	.1934	.9212	.0788	.1147	.8853	10′
40° 00′	9.8081	10.1919	9.9238	10.0762	10.1157	9.8843	50° 00′
10′	.8096	.1904	.9264	.0736	.1168	.8832	50′
20′	.8111	.1889	.9289	.0711	.1179	.8821	40′
30′	.8125	.1875	.9315	.0685	.1190	.8810	30′
40′	.8140	.1860	.9341	.0659	.1200	.8800	20′
50′	.8155	.1845	.9366	.0634	.1211	.8789	10′
41° 00′	9.8169	10.1831	9.9392	10.0608	10.1222	9.8778	49° 00′
	L cos θ	L sec θ	L cot θ	L tan θ	L csc θ	L sin θ	Angle θ

TABLE IV—*continued*

Attach − 10 to Logarithms Obtained from This Table

Angle θ	L sin θ	L csc θ	L tan θ	L cot θ	L sec θ	L cos θ	
41° 00′	9.8169	10.1831	9.9392	10.0608	10.1222	9.8778	**49° 00′**
10′	.8184	.1816	.9417	.0583	.1233	.8767	50′
20′	.8198	.1802	.9443	.0557	.1244	.8756	40′
30′	.8213	.1787	.9468	.0532	.1255	.8745	30′
40′	.8227	.1773	.9494	.0506	.1267	.8733	20′
50′	.8241	.1759	.9519	.0481	.1278	.8722	10′
42° 00′	9.8255	10.1745	9.9544	10.0456	10.1289	9.8711	**48° 00′**
10′	.8269	.1731	.9570	.0430	.1301	.8699	50′
20′	.8283	.1717	.9595	.0405	.1312	.8688	40
30′	.8297	.1703	.9621	.0379	.1324	.8676	30′
40′	.8311	.1689	.9646	.0354	.1335	.8665	20′
50′	.8324	.1676	.9671	.0329	.1347	.8653	10′
43° 00′	9.8338	10.1662	9.9697	10.0303	10.1359	9.8641	**47° 00′**
10′	.8351	.1649	.9722	.0278	.1371	.8629	50′
20′	.8365	.1635	.9747	.0253	.1382	.8618	40′
30′	.8378	.1622	.9772	.0228	.1394	.8606	30′
40′	.8391	.1609	.9798	.0202	.1406	.8594	20′
50′	.8405	.1595	.9823	.0177	.1418	.8582	10′
44° 00′	9.8418	10.1582	9.9848	10.0152	10.1431	9.8569	**46° 00′**
10′	.8431	.1569	.9874	.0126	.1443	.8557	50′
20′	.8444	.1556	.9899	.0101	.1455	.8545	40′
30′	.8457	.1543	.9924	.0076	.1468	.8532	30′
40′	.8469	.1531	.9949	.0051	.1480	.8520	20′
50′	.8482	.1518	9.9975	.0025	.1493	.8507	10′
45° 00′	9.8495	10.1505	10.0000	10.0000	10.1505	9.8495	**45° 00′**
	L cos θ	**L sec θ**	**L cot θ**	**L tan θ**	**L csc θ**	**L sin θ**	**Angle θ**

ANSWERS

Exercise 8.1

1. (i) $\sqrt{97}$ (ii) $2a\left|\sin\left(\frac{\alpha-\beta}{2}\right)\right|$ **4.** $5, -3$ **5.** $(3, 0)$ **6.** $x - y = 3$

Exercise 8.2

1. (a) $\frac{1}{2}$ (b) 18 (c) $\frac{75}{2}$ (d) $\frac{57}{2}$ **3.** $x = 1$ **4.** $(7, 2)$ and $(1, 0)$ **5.** $5x - 4y + 1 = 0$

Exercise 8.3

1. $((0, 0), (3, -9))$ **3.** $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$ **4.** $(-4, -15)$

5. $1 : 3$ **6.** $1 : 2$

Exercise 8.4

1. (a) Angle of inclination is acute.
   (b) Parallel to the x-axis or coincides with the x- axis.
   (c) Angle of inclination obtuse.

2. (a) 0 (b) 1 (c) undefined (d) $-\frac{1}{\sqrt{3}}$ **3.** (a) 0 (b) -1 (c) $-\frac{1}{6}$

5. (a) Parallel (b) Neither (c) Perpendicular (d) Parallel **6.** 1 **7.** 9

Exercise 8.5

1. $5x + 3y = 9$
2. $x^2 - 8x - 4y + 20 = 0$ **3.** $y = 3x$
5. $y = x + a$ where a is the given distance
6. $\frac{x^2}{5} + \frac{y^2}{9} = 1$ **7.** $y = \frac{2}{3}x$
8. $20x^2 + 36y^2 < 405$ **9.** $x^2 + y^2 + 2x - 4y - 4 > 0$

Exercise 9.1

1. $y = 2x - 5$
2. $y = -2(2x + 1)$
3. $5y = 3x - 15$
4. $y = -2$
5. $y = x\sqrt{3} + 2$, $y = -x\sqrt{3} + 2$; $y = \pm x\sqrt{3} - 2$; $\left(-\frac{2}{3}\sqrt{3}, 0\right)$, $\left(\frac{2}{3}\sqrt{3}, 0\right)$
6. 30°, 60° **7.** $3y = 2x + 12$ **8.** $3y = 5(x + 3)$
9. $2x + y = 6$ or $x + 2y = 6$

10. $y = 4,\ 2x - 3y = 0;\ \ 2x - y = 0$

11. $x = 2,\ 7y = 6x + 79,\ 7y = -(6x + 65)$

12. $5y = -2x + 18$ **13.** $3y = 2x - 7$

14. $x - y = 1$

16. (i) $y + x - 3\sqrt{2} = 0$
(ii) $y + \sqrt{3}x - 10 = 0$
(iii) $y - x + 5\sqrt{2} = 0$
(iv) $y = 1$

17. (i) $\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \sqrt{2}$
(ii) $-\frac{4}{5}x - \frac{3}{5}y + \frac{9}{5} = 0,\ \frac{9}{5}$
(iii) $x = 4,\ 4$ (iv) $y = 2,\ 2$

18. $x - 4y - 7 = 0$

19. $x + y + 5\sqrt{2} = 0$ **21.** 30°

22. $x = 0,\ \sqrt{3}y - x = 0,\ (0, -3),\ \left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

23. coincident **24.** Intersecting **25.** Parallel **26.** $\frac{16}{5}$ **27.** 2 **28.** 1

Exercise 10.1

1. (i) $2x + 29y = 0$
(ii) $13x - 19y = 83$
(iii) $x + 12y = 1$
(iv) $3x - 29y = 29$

2. (i) $42x + 21y = 257$
(ii) $21y = 113$
(iii) $7x = 24$
(iv) $63x + 105y = 781$

3. $15x + 12y = 7$ **4.** $6x - 17y = -24$

Exercise 10.2

1. $x = 2y,\ \ x = 3y$ **2.** $ax - by = 0$ and $bx + ay = 0$ **3.** $\tan^{-1}\frac{1}{3}$

4. $y = 3x,\ \ y + 2x = 0;$ 45°

Exercise 10.3

1. (i) $x^2 + xy = 0$
(ii) $xy - y^2 = 0$
(iii) $xy = 0$ (iv) $x^2 - y^2 = 0$

6. $2x = 4y + 1,\ \ 2x + y = 0$

Exercise 11.1

1. (i) $(0, 1),\ \sqrt{2}$ (ii) $(-5, 3),\ \sqrt{30}$

(iii) $(\frac{1}{2}, \frac{-1}{3}), \frac{1}{2}$ (iv) $(2, -3), 3\sqrt{2}$ (v) $(\frac{1}{2}, -1), \frac{\sqrt{17}}{2}$

2. (i) $36x^2 + 36y^2 - 36x - 18y + 11 = 0$
(ii) $x^2 + y^2 + 6x + 4y - 36 = 0$
(iii) $x^2 + y^2 + 2y = 0$
(iv) $x^2 + y^2 - x - y = 0$
(v) $x^2 + y^2 - 2ax - 2ay = 0$
(vi) $x^2 + y^2 - 2ax \cos \alpha - 2ay \sin \alpha = 0$

3. $x^2 + 4x + y^2 - 21 = 0$ and $x^2 + y^2 - 12x + 11 = 0$

Exercise 11.2

1. (i) $x^2 + y^2 + 6x + 2y - 90 = 0$
(ii) $x^2 + y^2 - 2x + 6y - 40 = 0$
(iii) $13x^2 + 13y^2 - 64x + 10y - 332 = 0$
(iv) $x^2 + y^2 + 4x + 6y - 12 = 0$

2. (a) $x^2 + y^2 - ax - by = 0$
(b) $x^2 + y^2 - 2hx - 2ky = p^2 + q^2 - 2hp - 2kq$

3. $x^2 + y^2 - 4x - 2y - 20 = 0$; $(2, 1)$, 5.

4. $x^2 + y^2 - 4x - 6y - 87 = 0$

Exercise 11.3

1. (i) $x = \frac{2}{\sqrt{3}} \cos \alpha, y = \frac{2}{\sqrt{3}} \sin \alpha$
(ii) $x = -1 + 3 \cos \alpha, \quad y = 2 + 3 \sin \alpha$
(iii) $x = -\frac{p}{2} + \frac{p}{\sqrt{2}} \cos \alpha, \quad y = -\frac{p}{2} + \frac{p}{\sqrt{2}} \sin \alpha$

2. (i) $x^2 + y^2 = 9$; $(0, 0)$, 3
(ii) $(x - a)^2 + (y - b)^2 = c^2$; (a, b), c
(iii) $(x - 7)^2 + (y + 3)^2 = 16$; $(7, -3)$, 4
(iv) $4x - y - 5 = 0$

Exercise 11.4

1. $x^2 + y^2 - (p + r)x - (q + s)y + pr + qs = 0$

2. $x^2 + y^2 - 3x - 2y - 21 = 0$

Exercise 11.5

1. $(2, -4)$ and $(4, 3)$

2. $\left[\frac{-mc \pm \sqrt{a^2(1 + m^2) - c^2}}{1 + m^2}, \frac{c \pm m\sqrt{a^2(1 + m^2) - c^2}}{1 + m^2}\right]$; $a^2(1 + m^2) = c^2$

3. $(0, 2)$ and $(2, 0)$ **4.** ± 5.

Exercise 11.6

1. (i) $x + y = 2$
   (ii) $x - 3y = 10$
   (iii) $3x + 4y = -25$
   (iv) $11x - 2y = 46$
   (v) $11x - 2y = 16$
   (vi) $4x + 3y + 6 = 0$
   (vii) $x \cos \alpha + y \sin \alpha = a(1 + \cos \alpha)$

3. $a^2(l^2 + m^2) = n^2$

4. $2x + y + 3\sqrt{5} = 0$ 5. $\frac{1}{2}\sqrt{46}$ 6. $4\sqrt{3}$ 8. $y = x\sqrt{3} \pm 2\sqrt{3}$

Exercise 11.7

1. $x^2 + y^2 - 4x + 2y + (5 - r^2) = 0$, $x^2 + y^2 - 4x + 2y - 4 = 0$
2. $x^2 + y^2 + 8x - 4y + (20 - r^2) = 0$, $2x^2 + 2y^2 + 16x - 8y - 41 = 0$
3. $x^2 + y^2 - 6x + 10y + (34 - r^2) = 0$, $x^2 + y^2 - 6x + 10y + 9 = 0$
4. (i) $6x^2 + 6y^2 - 44x + 43 = 0$
   (ii) $23x^2 + 23y^2 - 156x + 38y + 168 = 0$
5. (i) $x^2 + y^2 + 2gx - 6y + 9 = 0$, g is a parameter.
   (ii) $x^2 + y^2 + 8x + 2fy + 16 = 0$, f is a parameter.
6. $x^2 + y^2 + (3k - 4)x + 2ky + (3k - 4)^2 + k^2 - r^2 = 0$

Exercise 11.8

2. $3x^2 + 3y^2 - 14x + 23y - 15 = 0$

Exercise 12.1

1. (a) $(2, 0)$, $x = -2$ (b) $(0, \frac{3}{2})$, $y = -\frac{3}{2}$

   (c) $(-3, 0)$, $x = 3$ (d) $(0, -4)$, $y = 4$
2. (a) $y^2 = -16x$ (b) $x^2 = -8y$ (c) $2y^2 = 9x$
3. (a) $(1, \frac{9}{4})$, $(1, 2)$, $y = \frac{7}{4}$, $x = 1$

   (b) $(\frac{3}{8}, \frac{9}{16})$, $(\frac{3}{8}, \frac{1}{2})$, $y = \frac{5}{8}$, $x = \frac{3}{8}$

   (c) $(\frac{3}{2}, \frac{-15}{8})$, $(\frac{3}{2}, \frac{-11}{8})$, $y = -\frac{7}{8}$, $x = \frac{3}{2}$
4. (a) $y^2 = 12x - 36$ (b) $x^2 = 32 - 8y$
   (c) $4x^2 + 4xy + y^2 + 4x + 32y + 16 = 0$

Exercise 12.2

1. (a) $10, 8$, $(\pm 3, 0)$, $(\pm 5, 0)$, $\frac{3}{5}$

(b) $8, 6, (\pm\sqrt{7}, 0), (\pm 4, 0), \frac{\sqrt{7}}{4}$

(c) $2\sqrt{3}, 2\sqrt{2}, (0, \pm 1), (0, \pm\sqrt{3}), \frac{1}{\sqrt{3}}$

(d) $2, 1, (1 \pm \frac{\sqrt{3}}{2}, 0), (2, 0), (0, 0), \frac{\sqrt{3}}{2}$

2. (a) $9x^2 + 25y^2 = 225$ (b) $100x^2 + 36y^2 = 3600$
(c) $25x^2 + 9y^2 = 225$ (d) $7x^2 + 15y^2 = 247$
(e) $x^2 + 2y^2 = 18$ (f) $16x^2 + 7y^2 = 688$

3. $3x^2 + 4y^2 - 36x = 0$

4. $9x^2 + 5y^2 = 180$

Exercise 12.3

1. (a) $6, 8, \frac{5}{3}, (\pm 5, 0), (\pm 3, 0)$

(b) $2\sqrt{3}, 2\sqrt{2}, \sqrt{\frac{5}{3}}, (\pm\sqrt{5}, 0), (\pm\sqrt{3}, 0)$

(c) $\frac{2}{\sqrt{3}}, \sqrt{2}, \sqrt{\frac{5}{2}}, (\pm \sqrt{\frac{5}{6}}, 0), (\pm\frac{1}{\sqrt{3}}, 0)$

2. (a) $24x^2 - 25y^2 = 600$ (b) $9x^2 - 7y^2 - 343 = 0$ (c) $y^2 - x^2 = 5$

3. $15x^2 - y^2 = 15$ **4.** $2\sqrt{3}, 8, \sqrt{\frac{19}{3}}$

Exercise 12.4

3. $(2, 4)$ **5.** $yt = x + at^2$ **6.** $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

7. $y = 3x \pm \frac{31\sqrt{20}}{4\sqrt{93}}, (\mp\frac{9}{4}, \frac{\sqrt{20}}{\sqrt{93}}, \pm \frac{\sqrt{20}}{\sqrt{93}})$

8. $24y = 30x \pm\sqrt{161}$

Exercise 13.1

1. (a) $\frac{\pi}{12}$ (b) $\frac{-\pi}{8}$ (c) $\frac{17\pi}{9}$ (d) $\frac{7\pi}{3}$

2. (a) 14° 19′ (nearly) (b) − 114° 33′ (nearly) (c) 420° (d) 150°

3. $\frac{5\pi}{12}$ cm **4.** $\frac{20\pi}{3}$ cm **5.** 6π **6.** 25° 12′ **7.** (a) 11° 27′ 12″
(b) 18° 19′38″ (nearly) (c) 29° 47′ (nearly) **8.** 100°

Exercise 13.2

1. $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = -\sqrt{3}$, $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$, $\sec\theta = -2$ $\cot\theta = -\frac{1}{\sqrt{3}}$

2. $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\operatorname{cosec}\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

3. $\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\operatorname{cosec}\theta = -\frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = \frac{4}{3}$

Exercise 13.3

1. $\frac{220}{221}, \frac{171}{221}, \frac{220}{21}$ 13. $\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 2$

14. $\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}, -\sqrt{2}$ 15. $\sqrt{\frac{4+\sqrt{15}}{8}}, \sqrt{\frac{4-\sqrt{15}}{8}}, 4+\sqrt{15}$

16. $\sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{2\sqrt{2}}}$, $\sqrt{\frac{4+\sqrt{2}+\sqrt{6}}{2\sqrt{2}}}$, $-(\sqrt{2}+1)+\sqrt{4+2\sqrt{2}}$

Exercise 13.4

1.	(i) .9387	(ii) .7431	(iii) 1.402	(iv) 1.501
2.	(i) 32° 30′	(ii) 89° 30′	(iii) 88° 20′	(iv) 18° 20′
3.	(i) .5645	(ii) .4295	(iii) .9037	(iv) .9185
4.	(i) 31° 36′	(ii) 87° 34′	(iii) 38° 45′	(iv) 7° 49′

Exercise 13.7

1. $\theta = (2n+1)\frac{\pi}{4}$ or $2m\pi$, where $n, m \in I$.
2. $\theta = n\pi + (-1)^{n+1}\frac{\pi}{6}$ or $(2m+1)\frac{\pi}{2}$, where $n, m \in I$.
3. $\theta = \frac{n\pi}{2}$ or $\frac{m\pi}{2} - \frac{\pi}{8}$, where $n, m \in I$.
4. $\theta = n\pi$ or $2m\pi$, where $n, m \in I$.
5. $\theta = 2n\pi - \frac{\pi}{2}$ or $\frac{2m\pi}{5} - \frac{\pi}{10}$, where $n, m \in I$.
6. $x = 2n\pi + \frac{\pi}{4} \pm A$, where $n \in I$.
7. $\theta = n\pi + (-1)^n \sin^{-1}\left(\frac{-1+\sqrt{17}}{8}\right)$

   or $n\pi + (-1)^n \sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right)$, where $n \in I$.
8. $\theta = \frac{2k\pi}{m+n}$ or $\frac{(2k+1)\pi}{m-n}$, where $k \in I$.
9. $\theta = n\pi - \frac{\pi}{4}$ or $m\pi + \tan^{-1}\frac{1}{2}$, where $n, m \in I$.
10. $\theta = n\pi + (-1)^{n+1}\frac{\pi}{2}$ or $m\pi + (-1)^{m+1}\frac{\pi}{6}$, where $n, m \in I$.

Exercise 14.1

(i) $\frac{3}{5}, \frac{4}{5}, 1$

(ii) $\frac{3}{4}, \frac{4}{3}$, not defined

(iii) 216 sq. units

(iv) $\frac{1}{3}, \frac{1}{2}, 1$

Exercise 14.3

6. (i) $r = 6, A = 36° 52', B = 53° 8', C = 90°$
   (ii) $r = 1.5,\ A = 53° 8',\ B = 14° 14',\ C = 112° 38'$
8. 161.16 cm^2 (nearly), 307.92 cm^2 (nearly) and 1018.81 cm^2 (nearly)

Exercise 14.4

1. $B = 37°,\ b = 32$ (nearly), $c = 54$ (nearly)
2. $B = 64°, b = 5.64, a = 2.85$ (nearly)
3. $A = 62° 32',\ B = 27° 28',\ c = 56.8$
4. $A = 41°$ (nearly), $B = 49°$ (nearly), $a = 2.9$ (nearly)
5. $B = 51°, a = 3.6$ (nearly), $c = 5.8$ (nearly)
6. $A = 42° 29'\ B = 47° 31',\ b = 450$

Exercise 14.5

1. 58° 59′ 33″ (nearly) 2. $A = 45°, B = 120°, C = 15°$
3. 104° 28′ 39″ (nearly) 4. $A = 56° 15' 4'', B = 59° 51' 10'', C = 63° 53' 46''$

Exercise 14.6

1. $A = 117° 38' 45'',\ B = 27° 38' 45''$
2. $B = 71° 44' 30'',\ C = 48° 15' 30''$
3. $c = \sqrt{6},\ A = 105°,\ B = 15°$
4. $b = 40, A = 30°,\ C = 120°$

Exercise 14.7

1. There is no triangle.
2. $B_1 = 15°,\ C_1 = 135°,\ b_1 = 50(\sqrt{6} - \sqrt{2})$
   $B_2 = 105°,\ C_2 = 45°,\ b_2 = 50(\sqrt{6} + \sqrt{2})$
3. $c = 4\sqrt{3} \pm 2\sqrt{5}$

Exercise 14.8

1. (i) $a = 70$ (nearly), $b = 76$ (nearly), $C = 59°$
   (ii) $b = 40.5,\ c = 31.15,\ c = 47°$
   (iii) $a = 274$ (nearly), $c = 339$ (nearly), $B = 57° 50'$

Exercise 14.9

1. 18.12 m **2.** 17.32 m; 10 m **3.** 260.51 m **4.** 225 m **5.** 227.39 m
6. 68.3 m **7.** S 54° 9′ 43″ N; 22.2 km **8.** 146.4 m **9.** 17.32 m
10. 136.54 m, 1015.37 m **11.** 1366 m

Exercise 15.1

1. (i) $\frac{-\pi}{2}$ (ii) $\frac{-\pi}{6}$ (iii) $\frac{2\pi}{3}$ (iv) $\frac{\pi}{6}$ (v) $\frac{\pi}{6}$ (vi) $\frac{-\pi}{6}$

11. $\frac{x+y}{1-xy}$, where $xy < 1$ **12.** (i) $\pm\frac{1}{\sqrt{2}}$ (ii) $\frac{\pi}{4}$

13. (i) $\sin^{-1}\frac{x}{a}$ (ii) $\frac{x}{2}$ (iii) $\frac{\pi}{4} - x$ (iv) $3\tan^{-1}\frac{x}{a}$

Exercise 16.1

2. (a) 12.75

4. 38.9 paise, Rs 0.39. (See problem 5 for the explanation)

6. (a) 8.8 years, 8.2 years (b) The concentration of data in class IV of school *A* is much higher than that of class IV of school *B*.

7. The mean age is 23.6 years approximately.

8. 40.6 mm (approx.)

Exercise 16.2

1. Median = 74th observation = 12. Thus, for about half the number of days, more than 12 students were absent.

2. Median = $\frac{23+24}{2}$ = 23.5. About 50% of the students obtained less than 23.5 marks out of 50 in the test.

3. First note that the class 15-19 represents "15 years or more but less than 20". Median = 24.5 years. Nearly half the women were married between the ages 15 and 24.5.

4. The distribution of data is not symmetrical.

Exercise 16.3

2. (a) The variance is the same in both the cases and equal to the variance of the original scores obtained.
(b) 112.64

4. (a) 69.16
(b) Variance in case (b) is 4 times the variance in case (a). (See Problem 5 of this exercise for the reason.)

6. $\sigma^2 = 140.70$, $\bar{x} - \sigma = 78.58$, $\bar{x} + \sigma = 102.30$, required percent = 64. You may subtract 92 from each score to make calculations easier.

Exercise 16.4

1. $M = \dfrac{49.4 + 49.6}{2} = 49.5$ cm, $d = 21.5$ cm, 27.
2. $M = 67.1$ m^2, $d = 20.9$ m^2, 48%, 97%

Exercise 17.1

1. Polygon with vertices (0, 0), (3, 0), $\left(\frac{12}{5}, \frac{2}{5}\right)$, (0, 1)
2. Polygon with vertices (1, 0), (6, 0), $(6, \frac{7}{3})$, $(\frac{18}{7}, 5)$, (0, 5), (0, 1)
5. $x \geq 0, \quad 2x + y \geq 2, x - y \leq 1, x + 2y \leq 8, \; y \geq 0$
6. $x \geq 0, \; 2x + 3y \geq 3, \; y \geq 0, \; x - 6y \leq 3, 3x + 4y \leq 18, \; -7x + 4y \geq 14$

Exercise 17.2

1. The set of feasible solutions is a polygon with vertices $\left(\frac{18}{7}, \frac{2}{7}\right)$, $\left(\frac{7}{2}, \frac{3}{4}\right)$, $\left(\frac{3}{2}, \frac{15}{4}\right)$, $\left(\frac{3}{13}, \frac{24}{13}\right)$. Hence, max. = 19, min. = $\frac{63}{13}$.
2. Max. = $14\frac{1}{13}$ is at $\left(\frac{84}{13}, \frac{15}{13}\right)$, and Min. = $3\frac{1}{14}$ is at $\left(\frac{9}{14}, \frac{25}{14}\right)$.
3. Even though the set of feasible solutions is not a closed polygon the minimum can still be found by the graphical method. Min. $9\frac{2}{3}$ is at $\left(\frac{8}{3}, \frac{1}{3}\right)$.
4. Let x km = distance travelled at 25 km/h
   y km = distance travelled at 40 km/h.
   We maximise $x + y$ subject to the constraints
   $$2x + 5y \leq 100,$$
   $$\frac{x}{25} + \frac{y}{49} \leq 1,$$
   $$x, y \geq 0.$$
   Max. = 30 km with $x = \frac{50}{3}, y = \frac{40}{3}$
5. If x, y denote quantities (in grams) of wheat and rice, our problem is to minimise $\frac{2x}{1000} + \frac{8y}{1000}$ subject to the constraints
   $$0.1x + 0.05y \geq 50$$
   $$0.25x + 0.5y \geq 200$$
   $$x, y \geq 0$$
   Diet cost is minimum (Rs 2.40) when $x = 400$ g, $y = 200$ g.

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